

A complete waveform model for compact binaries on eccentric orbits

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We present a time domain waveform model that describes the inspiral, merger and ringdown of compact binary systems whose components are non-spinning, and which evolve on orbits with low to moderate eccentricity. The inspiral evolution is described using third order post-Newtonian equations both for the equations of motion of the binary, and its far-zone radiation field. This latter component also includes instantaneous, tails and tails-of-tails contributions, and a contribution due to non-linear memory. This framework reduces to the post-Newtonian approximant **TaylorT4** at third post-Newtonian order in the zero eccentricity limit. To improve phase accuracy, we also incorporate higher-order post-Newtonian corrections for the energy flux of quasi-circular binaries and gravitational self-force corrections to the binding energy of compact binaries. This enhanced prescription for the inspiral evolution is combined with a fully analytical prescription for the merger-ringdown evolution constructed using a catalog of numerical relativity simulations. We show that this inspiral-merger-ringdown waveform model reproduces the effective-one-body model of Ref. [1] for black hole binaries with mass-ratios between 1 to 15 in the zero eccentricity limit over a wide range of the parameter space under consideration. We use this model to show that the gravitational wave transients GW150914 and GW151226 can be effectively recovered with template banks of quasi-circular, spin-aligned waveforms if the eccentricity e_0 of these systems when they enter the aLIGO band at a gravitational wave frequency of 14 Hz satisfies $e_0^{\text{GW150914}} \leq 0.15$ and $e_0^{\text{GW151226}} \leq 0.1$. We also show that quasi-circular, spin-aligned compact binaries define a signal manifold that is predominantly orthogonal to a manifold generated by non-spinning, eccentric signals for $e_0 \geq 0.1$.

I. INTRODUCTION

The field of gravitational wave (GW) astronomy has been firmly inaugurated with the first direct detections of gravitational radiation from binary black hole (BBH) systems with the Advanced Laser Interferometer Gravitational-wave Observatory (aLIGO) detectors [2–4]. The growing sample of BBH observations that is expected in aLIGO’s next observing runs [4, 5] will enable an accurate census of the mass and angular momentum distribution of BHs and neutron stars (NSs), gaining insights into formation and evolution scenarios of compact object binaries, and the environments in which they reside [6–10]. For instance, the detection of GWs from eccentric compact binaries can provide important information of compact object populations in globular clusters and galactic nuclei [7]. Any such analysis must start with the development of waveforms for eccentric compact binaries, the topic of this article.

GWs encode information about the properties of the astrophysical sources that generate them, and can be used to map the structure of spacetime in the vicinity of compact binary systems [11]. aLIGO is expected to detect a wide variety of GW sources, including: (i) compact binary systems that form in the galactic field and evolve through massive stellar evolution. These are expected to enter aLIGO’s frequency band on nearly quasi-circular orbits because GWs are very effective at circularizing the orbits of compact binaries [12, 13]; (ii) compact binaries formed in dense stellar environments, e.g., core-collapsed globular clusters and galactic nuclei. In these environments, compact systems can undergo a variety of N-body interactions that lead to the formation of compact binaries that retain eccentricity during their lifetime (see [7, 10, 14–17] and references therein).

The detection of stellar mass BHs in the galactic cluster M22 [15] led to the development of more accurate N-body algorithms to explore the formation and detectability of BBHs formed in globular clusters with aLIGO. These improved analyses indicate that about 20% of BBH mergers in globular clusters will have eccentricities

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$e_0 \gtrsim 0.1$ when they first enter aLIGO band at 10Hz, and that $\sim 10\%$ may have eccentricities $e \sim 1$ [7]. Furthermore, a fraction of galactic field binaries may retain significant eccentricity prior to the merger event [18]. BBHs formed in the vicinity of supermassive BHs may also merge with significant residual eccentricities [19]. Given the proven detecting capabilities of aLIGO, these results imply that we are now in a unique position to enhance the science reach of GW astronomy by targeting eccentric compact binary systems. The detection of these events requires the development of new waveform models and data analysis techniques because the imprint of eccentricity on GWs is multifold: it introduces modulations in the amplitude and frequency evolution of the waveforms, and it shortens their duration [20–31]. GWs emitted by compact binaries that enter aLIGO band with moderate eccentricities, $e_0 \lesssim 0.4$, can be modeled as continuous waves and searched for using matched-filtering algorithms. In contrast, systems that enter aLIGO band with $e_0 \sim 1$ emit individual GW bursts at each periastron passage, most suitably searched by excess power algorithms utilizing time-frequency tiling [32].

In order to detect and characterize eccentric binary systems with aLIGO, this paper introduces the first inspiral-merger-ringdown (IMR) waveform model that is adequate for source detection and parameter estimation, under the assumption that the binary components are non-spinning. This model can be immediately used in the context of aLIGO to: (i) quantify the sensitivity of quasi-circular searches and burst searches to eccentric signals; (ii) study template bank construction for non-spinning, eccentric BBHs; (iii) estimate the eccentricity of detected BBH signals, under the assumption that the binary components are not spinning; (iv) explore the sensitivity of burst-like searches that have been tuned to detect highly eccentric systems ($e_0 \sim 1$) to recover signals with moderate values of eccentricity, i.e., $e_0 \leq 0.4$ [33–38].

Previous work related to this particular subject includes: (i) Frequency domain inspiral-only waveforms that include leading order PN¹ corrections in a post-circular or small eccentricity approximation [39, 40]. (ii) Frequency and time domain waveforms that reduce to the PN-based approximants **TaylorF2** and **TaylorT4** at 2PN in the quasi-circular limit [41]. (iii) Inspiral-only waveforms that include 2PN and 3PN corrections to the radiative and conservative pieces of the dynamics, respectively [42]. (iv) Inspiral-only waveforms that include 3PN corrections to the radiative and conservative pieces of the dynamics [28, 43–45]. (v) Inspiral-only frequency domain waveforms that reduce to the PN-based approximant **TaylorF2** 3.5PN at zero eccentricity, and to the post-circular approximation of Ref. [39] at small eccen-

tricity [31]. (vi) Hybrid waveforms that describe highly eccentric systems. These waveforms describe the inspiral evolution using geodesic equations of motion. The merger phase is modeled using a semi-analytical prescription that captures the features of numerical relativity simulations [46]. (vii) Self-force calculations for non-spinning BHs along eccentric orbits [47–54]. (viii) Numerical relativity simulations that explore the dynamics of eccentric binary systems [42, 55–61].

Some of the aforementioned waveform models have been used in source detection [30, 31, 62, 63] and parameter estimation studies [64, 65] in the context of aLIGO. These studies have shown that detecting and characterizing eccentric binary systems will not be feasible using existing algorithms for quasi-circular binaries [30, 64]. Furthermore, as discussed in [31], to accurately model inspiral-dominated systems, i.e., binary systems with total mass $M \lesssim 10M_\odot$ [66], eccentric waveform models should reduce to high PN order approximants such as **TaylorT4** 3.5PN or **TaylorF2** 3.5PN [30, 31] in the zero eccentricity limit. On the other hand, for NSBH and BBH systems that require the inclusion of the merger and ringdown phase, eccentric waveform models should reproduce the evolution rendered by IMR models such as [1, 67, 68] in the zero eccentricity limit.

In this paper we start addressing these important issues by developing an IMR waveform model valid for compact binaries with moderate eccentricities $e_0 \lesssim 0.4$. The key features of our model are:

- It includes third order PN accurate expansions for eccentric orbits both for the equations of motion of the binary and its far-zone radiation field. The radiative evolution includes instantaneous, tails and tails-of-tails contributions, and a contribution due to non-linear memory.
- The accuracy of the inspiral evolution is improved by including 3.5PN corrections for quasi-circular orbits (at all powers of symmetric mass-ratio), improving on [31].
- To further improve phase accuracy especially for unequal mass systems, the 3PN accurate inspiral evolution for eccentric systems is corrected by including up to 6PN terms both for the energy flux of quasi-circular binaries and gravitational self-force corrections to the binding energy of compact binaries at first order in symmetric mass-ratio η .
- We combine the aforementioned enhanced inspiral evolution with a merger and ringdown treatment using the *implicit rotating source* (IRS) formalism [69], fitted against numerical relativity simulations up to mass-ratio 10.

The eccentric model we develop in this article is the first model in the literature that combines all these features, and makes it a powerful tool to explore the detection of eccentric signals with aLIGO. To exhibit

¹ When we state the accuracy of post-Newtonian (PN) expansions below, a term of Nth PN order implies that the term of highest order in the weak-field expansion is proportional to $(v/c)^{2N}$, where v represents the orbital velocity [26].

the reliability of our eccentric model, we show that it agrees well with the IMR effective-one-body model SEOBNRv2 [85, 86] in the non-spinning limit over a wide range of the BBH parameter space accessible to aLIGO. Having established the validity of our new eccentric model, we use it to shed light for the first time on the importance of including eccentricity in the detection of IMR systems, such as NSBH and BBH systems with asymmetric mass-ratios. We also show that our waveform model has a favorable computational cost, suitable for large scale data analysis studies.

Throughout this article we use units $G = c = 1$. We denote the components masses by m_1 and m_2 , where $m_1 \geq m_2$. Mass combinations used throughout the article include: total mass $M = m_1 + m_2$, reduced mass $\mu = m_1 m_2 / M$, mass-ratio $q = m_1 / m_2$, and symmetric mass-ratio $\eta = \mu / M$.

This paper is organized as follows: In Section II we describe the construction of our eccentric waveform model. In Section III we apply our eccentric waveform model to explore the detectability of eccentric compact binary systems with aLIGO. We summarize our results and discuss future directions of work in Section IV.

II. WAVEFORM MODEL CONSTRUCTION

A. Overview

In this Section we introduce our new eccentric waveform model, named advanced x -model or ‘ ax -model’, since it extends the inspiral-only, low order PN eccentric x -model introduced in [42]. The construction of our model has several key ingredients that are described on an incremental basis.

In the description below we refer to the conservative and radiative pieces of the dynamics. The conservative piece refers to the equations of motion of the binary that are derived from a PN Hamiltonian [26], whereas the radiative piece takes into account the energy and angular momentum that gravitational radiation carries away from the binary.

B. Eccentric orbit parameterization

We parametrize the equations of motion in terms of the mean orbital frequency ω through the gauge invariant quantity $x = (M\omega)^{2/3}$, and the *temporal* eccentricity $e_t \equiv e$ [42]. Please note that in the context of eccentric binaries, $\omega = \langle \dot{\phi} \rangle = Kn$, the mean motion n is related to the mean anomaly ℓ through the relation $M\dot{\ell} = Mn$ (see Eq. 3 below), $\dot{\phi}$ is the instantaneous angular velocity, and the periastron precession K and relativistic precession k are related through $K = 1 + k$. At 3PN order, the Keplerian parameterization of the orbit in terms of the magnitude of the relative separation vector r , and the

mean anomaly ℓ is given by [42]:

$$\frac{r}{M} = \frac{1 - e \cos u}{x} + \sum_{i=1}^{i=3} r_{i\text{PN}} x^{i-1}, \quad (1)$$

$$\ell = u - e \sin u + \sum_{i=2}^{i=3} l_{i\text{PN}} x^i. \quad (2)$$

The orbital evolution has two components. The conservative piece is derived from a PN Hamiltonian including corrections at 3PN order and has the form:

$$M\dot{\phi} = \dot{\phi}_{0\text{PN}} x^{3/2} + \dot{\phi}_{1\text{PN}} x^{5/2} + \dot{\phi}_{2\text{PN}} x^{7/2} + \dot{\phi}_{3\text{PN}} x^{9/2} + \mathcal{O}(x^{11/2}), \quad (3)$$

$$M\dot{\ell} = Mn = x^{3/2} + n_{1\text{PN}} x^{5/2} + n_{2\text{PN}} x^{7/2} + n_{3\text{PN}} x^{9/2} + \mathcal{O}(x^{11/2}), \quad (4)$$

where ϕ represents the relative orbital phase. The PN coefficients ($r_{i\text{PN}}$, $l_{i\text{PN}}$), ($\dot{\phi}_{i\text{PN}}$, $n_{i\text{PN}}$) are given in [42]. The radiative part of the orbital evolution takes into account the energy and angular momentum that gravitational radiation carries away from coalescing compact binaries. This effect implies that the gauge-invariant expansion parameter x and the eccentricity e are no longer conserved, but evolve as follows:

$$M\dot{x} = \dot{x}_{0\text{PN}} x^5 + \dot{x}_{1\text{PN}} x^6 + \dot{x}_{2\text{PN}} x^7 + \dot{x}_{3\text{PN}} x^8 + \dot{x}_{\text{HT}}, \quad (5)$$

$$M\dot{e} = \dot{e}_{0\text{PN}} x^4 + \dot{e}_{1\text{PN}} x^5 + \dot{e}_{2\text{PN}} x^6 + \dot{e}_{3\text{PN}} x^7 + \dot{e}_{\text{HT}}. \quad (6)$$

In the above expressions we have derived 3PN corrections for \dot{x} , and have also derived hereditary terms (HT) \dot{x}_{HT} . These new calculations are presented in Appendix A. Hereditary terms are non-linear contributions that depend on the dynamics of the system in its entire past, and comprise tails, tails-of-tails and tail square terms for the energy and angular flux, but also a 2.5PN memory contribution for the angular momentum flux. These terms include fractional powers in x — see Equations (A7) and (A21). Regarding the time evolution of the eccentricity e , we use 3PN calculations and the corresponding hereditary contributions derived in Ref. [28]. In constructing this model, we have ensured that the choice of coordinates is consistent throughout, i.e., we are using modified harmonic coordinates. We construct the PN waveform strain as follows:

$$h^{\text{inspiral}}(t) = h_+^{\text{inspiral}}(t) - i h_\times^{\text{inspiral}}(t), \quad (7)$$

with the plus and cross polarizations given by [42]:

$$h_+ = -\frac{M\eta}{R} \left\{ (\cos^2 \iota + 1) \left[\left(-\dot{r}^2 + r^2 \dot{\phi}^2 + \frac{M}{r} \right) \cos 2\Phi \right. \right. \\ \left. \left. + 2r\dot{r}\dot{\phi} \sin 2\Phi \right] + \left(-\dot{r}^2 - r^2 \dot{\phi}^2 + \frac{M}{r} \right) \sin^2 \iota \right\}, \quad (8)$$

$$h_\times = -\frac{2M\eta}{R} \cos \iota \left\{ \left(-\dot{r}^2 + r^2 \dot{\phi}^2 + \frac{M}{r} \right) \sin 2\Phi \right. \\ \left. - 2r\dot{r}\dot{\phi} \cos 2\Phi \right\}, \quad (9)$$

where $\Phi = \phi - \chi$, and (χ, ι) represent the polar angles of the observer, and R the distance to the binary.

C. Eccentricity decay

In this Section we explore the importance of including 3PN accurate eccentricity corrections to the binary evolution. To do so we consider a population of BBH systems with component masses $m_{1,2} \in [5M_\odot, 50M_\odot]$, and with an orbital eccentricity e_0 when they enter aLIGO frequency band at a GW frequency $f_{\text{GW}} = 15\text{Hz}$.

Figure 1 presents the residual eccentricity at the last stable circular orbit (ISCO) for the aforementioned BBH population using equations of motion that include conservative and radiative corrections up to 3PN order (cf. Eq. (5)) at the ISCO frequency given by [39]

$$f_{\text{ISCO}} = \frac{1}{\pi M} \left(\frac{1+e}{6+2e} \right)^{3/2}. \quad (10)$$

Figure 1 includes contour lines of residual eccentricity at ISCO, namely: $e_{\text{ISCO}} = \{0.01, 0.02, 0.05, 0.1\}$. A key assumption in the construction of our eccentric model is that moderately eccentric binaries attain circularization prior to the merger event. In practice, we consider compact binary systems whose residual eccentricity at ISCO satisfies $e_{\text{ISCO}} \lesssim 0.05$. Figure 1 indicates that this assumption covers a wide range of the parameter space for moderately eccentric systems. We note that $e_0 = 0.4$ at $f_{\text{GW}} = 15\text{Hz}$ is already a very high value for astrophysically motivated systems.

Figure 1 also indicates that the largest value of residual eccentricity in all cases corresponds to the most massive BBH systems under consideration, which merge at lower frequencies and have less time to circularize under GW emission. On the other hand, BBH systems with less massive components merge at higher frequencies, and therefore undergo further circularization within the aLIGO frequency band.

In the previous study [31] we emphasized the importance of developing waveform models that encode higher-order PN corrections. We showed that waveform templates that include only 2PN corrections for the radiative

piece of the dynamics will significantly reduce the ability to observe eccentric compact binaries. To further explore the effect of including higher-order PN corrections, Figure 2 presents the difference in the number of GW cycles \mathcal{N} when we use a waveform model that includes conservative corrections up to 3PN order and radiative corrections up to 2PN or 3PN order. \mathcal{N} is defined as

$$N = \frac{1}{\pi} \left[\langle \phi \rangle (f_{\text{ISCO}}) - \langle \phi \rangle (f_{\text{min}}) \right], \quad (11)$$

and $f_{\text{min}} = 15\text{Hz}$. The color bar in Figure 2 describes $\Delta\mathcal{N} = \mathcal{N}(3\text{PN}) - \mathcal{N}(2\text{PN})$. These results demonstrate that waveform templates using only 2PN radiative corrections will significantly deviate from waveform models that include all known eccentric corrections up to 3PN order when $e_0 \gtrsim 0.2$, particularly for asymmetric mass-ratio systems.

In summary, the results of this Section indicate that an astrophysically motivated population of moderately eccentric compact binaries will circularize prior to the merger event. For these systems, it is physically motivated to add a non-eccentric merger waveform to the inspiral evolution. Finally, we have discussed the importance of including all known eccentric PN calculations to provide the most accurate description of these systems.

D. Improved non-eccentric terms

The inspiral evolution of the *ax*-model reduces to the PN-based approximant **TaylorT4** 3PN in the quasi-circular limit. To explicitly show this feature, we simplify the equations we derived in Appendix A in the $e \rightarrow 0$ limit. Please note that to obtain the following results it is necessary to include the hereditary corrections presented in equation (5), since these cancel out gauge-dependent quantities that are present in the instantaneous part of the fluxes. To be precise, this cancellation takes place because we include the tails-of-tails contributions in the fluxes. After including these non-linear contributions, equation (5) takes the form:

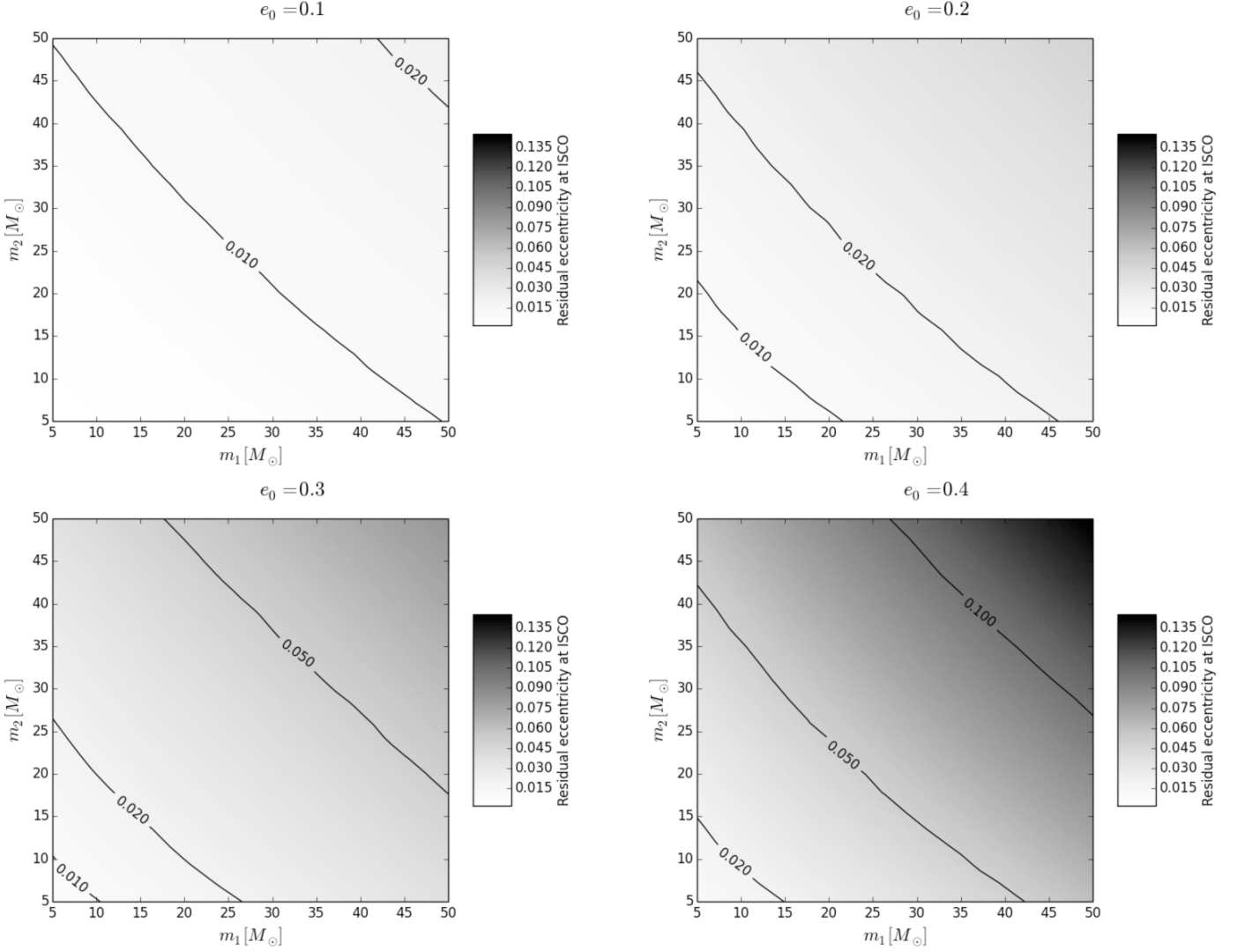


FIG. 1. The panels show the eccentricity at ISCO of black hole binaries prior to the merger event. We assume that the black hole population on each panel has an initial eccentricity e_0 at a gravitational wave frequency $f_{\text{GW}} = 15\text{Hz}$. These results have been obtained using PN equations of motion that include conservative and radiative corrections up to 3PN order.

$$\begin{aligned}
 M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{3\text{PN}} = & \frac{64}{5} \eta x^5 \left\{ 1 + \left(-\frac{743}{336} - \frac{11}{4} \eta \right) x + 4\pi x^{3/2} \right. \\
 & + \left(\frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2 \right) x^2 \\
 & + \left(-\frac{4159\pi}{672} - \frac{189\pi}{8} \eta \right) x^{5/2} \\
 & + \left[\frac{16447322263}{139708800} - \frac{1712\gamma_E}{105} + \frac{16\pi^2}{3} \right. \\
 & - \frac{856}{105} \log(16x) + \left(-\frac{56198689}{217728} + \frac{451\pi^2}{48} \right) \eta \\
 & \left. \left. + \frac{541}{896} \eta^2 - \frac{5605}{2592} \eta^3 \right] x^3 \right\}. \quad (12)
 \end{aligned}$$

Furthermore, the equations of the time evolution of the eccentricity e , relative orbital phase ϕ , and the mean anomaly ℓ reduce to [28, 42]:

$$M \frac{d\phi}{dt} \Big|_{e \rightarrow 0} = x^{3/2}, \quad (13)$$

$$M \frac{de}{dt} \Big|_{e \rightarrow 0} = 0, \quad (14)$$

$$\begin{aligned}
 M \frac{d\ell}{dt} \Big|_{e \rightarrow 0} = & x^{3/2} \left\{ 1 + 3x + \left(7\eta - \frac{9}{2} \right) x^2 \right. \\
 & \left. + \left(-\frac{27}{2} + \left(\frac{481}{4} - \frac{123}{32} \pi^2 \right) \eta - 7\eta^2 \right) x^3 \right\}. \quad (15)
 \end{aligned}$$

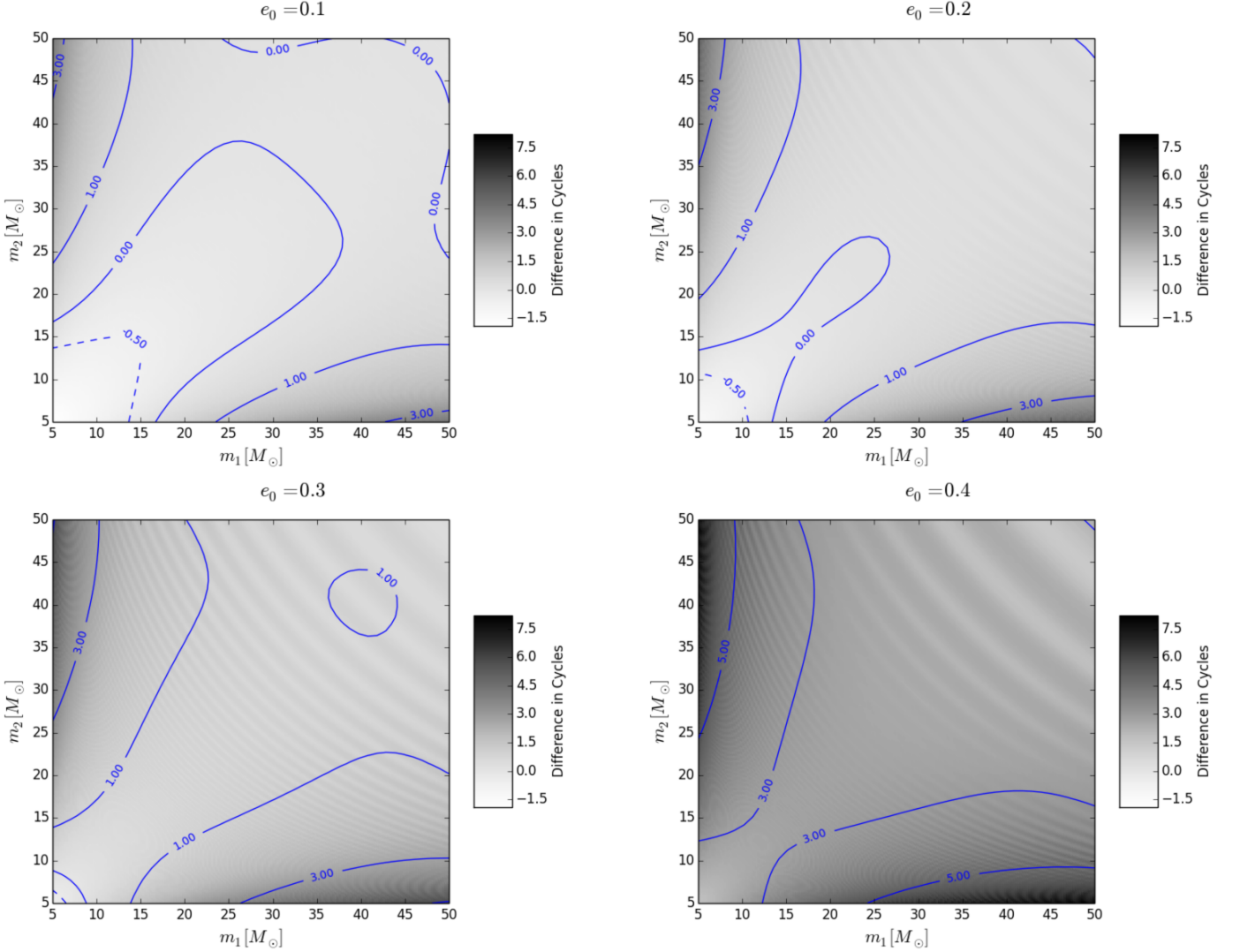


FIG. 2. The panels show the difference in the number of GW cycles when we use a waveform model that includes conservative corrections up to 3PN order and radiative corrections up to 2PN or 3PN order. The binary black hole population on each panel has an initial eccentricity e_0 at $f_{\text{GW}} = 15\text{Hz}$. Note that the discrepancy between the two approximate models becomes very noticeable when $e_0 \gtrsim 0.3$ for systems with asymmetric mass-ratios.

Note that Equation (15) describes the periastron advance in the $e \rightarrow 0$ limit. In order to further increase the reliability of our waveform model for inspiral dominated systems, we include 3.5PN corrections to the radiative equations of motion in the quasi-circular limit:

$$M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{3.5\text{PN}} = \frac{dx}{dt} \Big|_{e \rightarrow 0}^{3\text{PN}} + \frac{64\pi}{5} \eta x^5 \left[-\frac{4415}{4032} + \frac{358675}{6048} \eta + \frac{91945}{1512} \eta^2 \right] x^{7/2}, \quad (16)$$

where the first term on the right hand side of Eq. (16) is given by Eq. (12). Several studies argue that 3.5PN corrections are not sufficient for many applications, such as parameter estimation [66, 71, 72]. Therefore, to im-

prove phase accuracy for asymmetric mass-ratio systems, in this article we use the energy flux, $\dot{E}^{6\text{PN}}(x, \eta)$, derived in Ref. [74] up to 6PN order and amend it by including all known finite mass-ratio corrections for the energy flux of quasi-circular compact binaries. We then combine this prescription for the energy flux with the 6PN expression for the binding energy $E(x, \eta)^{6\text{PN}}$ of compact systems derived in [75, 76]², i.e.,:

² The results presented in this Section were computed using the expressions for the binding energy of Refs. [75, 76]. We found that the results from both prescriptions rendered very similar results. In the rest of this paper we quote results obtained using $E(x, \eta)^{6\text{PN}}$ from Ref. [75].

$$\begin{aligned}
M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{6\text{PN}} &= M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{3.5\text{PN}} + M \dot{E}^{6\text{PN}} \frac{dx}{dE(x, \eta)^{6\text{PN}}}, \quad (17) \\
M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{6\text{PN}} &= M \frac{dx}{dt} \Big|_{e \rightarrow 0}^{3.5\text{PN}} + \frac{64\eta x^5}{5} \left[a_4 x^4 + a_{9/2} x^{9/2} \right. \\
&\quad \left. + a_5 x^5 + a_{11/2} x^{11/2} + a_6 x^6 \right], \quad (18)
\end{aligned}$$

and the coefficients a_4 , $a_{9/2}$, a_5 , $a_{11/2}$, a_6 are presented in Appendix B. We have found that a model that combines Eq. (18) with the merger-ringdown model presented in the following Section agrees well with SEOBNRv2 up to mass-ratios $q = 15$. We present a quantitative discussion of this result in Section II H.

E. Merger and Ringdown evolution

We now turn our attention to the late time dynamical evolution. To construct the merger phase of our ax -model, we assume that the system circularizes prior to the merger event, i.e., the eccentricity at ISCO $e_{\text{ISCO}} \lesssim 0.05$. Under this assumption, we complement the inspiral evolution of the ax -model with a non-eccentric merger waveform. This stand alone merger waveform is constructed by calibrating the IRS model introduced by Kelly *et al* [69] with a catalog of NR simulations [79] obtained with the Spectral Einstein Code [78]. These simulations describe non-spinning, quasi-circular compact binary systems with mass-ratios between $q = 2.5$ and $q = 10$ [79, 80]. To ensure that our merger waveform reproduces the expected behavior of extreme-mass ratio binaries, we also utilize an SEOBNRv2 waveform with mass-ratio $q = 1000$, since the SEOBNRv2 model is tuned to black hole perturbation theory calculations. The IRS model encapsulates the evolution of the orbital frequency evolution, $\omega(t)$, and the waveform amplitude, $A(t)$, using the prescription [46, 69, 77, 81]:

$$\omega(t) = \omega_{\text{QNM}} (1 - \hat{f}), \quad (19)$$

$$\omega_{\text{QNM}} = 1 - 0.63 (1 - \hat{s}_{\text{fin}})^{0.3}, \quad (20)$$

$$A(t) = \frac{A_0}{\omega(t)} \left[\frac{|\dot{\hat{f}}|}{1 + \alpha (\hat{f}^2 - \hat{f}^4)} \right], \quad (21)$$

where \hat{s}_{fin} is the spin of the BH remnant. Furthermore, \hat{f} and $\dot{\hat{f}}$ are given by [46]:

$$\hat{f} = \frac{c}{2} \left(1 + \frac{1}{\kappa} \right)^{1+\kappa} \left[1 - \left(1 + \frac{1}{\kappa} e^{-2t/b} \right)^{-\kappa} \right], \quad (22)$$

$$\dot{\hat{f}} = \frac{d\hat{f}}{dt}. \quad (23)$$

Using the NR catalog previously described, we have derived the following analytical fit for \hat{s}_{fin} :

$$\hat{s}_{\text{fin}} = 2\sqrt{3}\eta - \frac{390}{79}\eta^2 + \frac{2379}{287}\eta^3 - \frac{4621}{276}\eta^4. \quad (24)$$

This prescription for \hat{s}_{fin} reproduces NR results with an accuracy better than 0.02%. Using this prescription, Eq. (20) is fully determined. The other free parameters in Eq. (19) are b , c and κ . Please note that in previous studies with the IRS model, these parameters have been determined for a few mass-ratio values using a catalog of NR simulations [46]. In this article, we develop a merger waveform that is reliable for systems with mass-ratios up to $q = 10$, and which also reproduces the correct behavior of extreme-mass ratio binaries. *The novelty of our approach is that we now provide the free parameters b , c and κ as smooth functions of the symmetric mass-ratio η .* To do this, we have used five NR simulations with mass-ratios $q = \{2.5, 4.5, 6.5, 9, 10\}$ and an SEOBNRv2 waveform of mass-ratio $q = 1000$. Using these waveforms as input data, we have constructed the following functions:

$$b(\eta) = \frac{16014}{979} - \frac{29132}{1343}\eta^2, \quad (25)$$

$$c(\eta) = \frac{206}{903} + \frac{180}{1141}\sqrt{\eta} + \frac{424}{1205} \frac{\eta^2}{\log(\eta)}, \quad (26)$$

$$\kappa(\eta) = \frac{713}{1056} - \frac{23}{193}\eta. \quad (27)$$

Note that these expressions are well behaved throughout the whole range of the symmetric mass-ratio η . Turning to the amplitude $A(t)$, see Eq. (21), we need to fix the extra parameter α . Following the approach outlined above we have found the parameterization:

$$\alpha(\eta) = \frac{1}{Q^2(\hat{s}_{\text{fin}})} \left(\frac{16313}{562} + \frac{21345}{124}\eta \right), \quad (28)$$

$$Q(\hat{s}_{\text{fin}}) = \frac{2}{(1 - \hat{s}_{\text{fin}})^{0.45}}. \quad (29)$$

Note that Eq. (28) is an extension to all symmetric mass-ratio η values of the fit quoted in [46]. The fit for the quality factor $Q(\hat{s}_{\text{fin}})$ in Eq. (29) was proposed in Ref. [81]. Having determined the analytical expressions for the free parameters $b(\eta)$, $c(\eta)$, $\kappa(\eta)$ and $\alpha(\eta)$, we are equipped to provide a robust description of the merger phase for compact binaries with non-spinning components and mass-ratios $q \leq 10$. *Since this framework enables us to describe in a unified framework the merger of non-spinning compact binaries over a wide range of mass-ratios, we label this formalism ‘generic IRS’ (gIRS) model.* In the left panel of Figure 3 we show the suite of numerical simulations used to obtain Eqs. (25)–(27). The right panel of Figure 3 shows that this simple prescription accurately reproduces the evolution of NR waveforms that were *not*

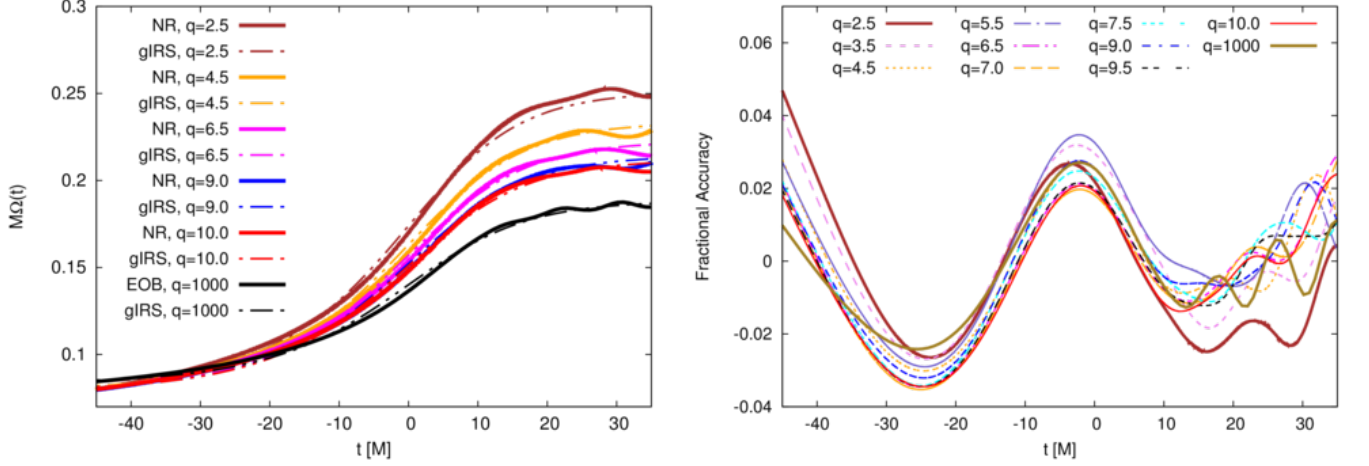


FIG. 3. Left panel: time evolution of the orbital frequency evolution, $M\Omega(t)$, of NR simulations and an SEOBNRv2 waveform ($q = 1000$) compared with our gIRS model, Eq. (19). The right panel shows that the analytical expressions given by Eqs. (25)–(27) can accurately reproduce the orbital frequency evolution of NR waveforms at mass-ratios that were *not* used for their calibration.

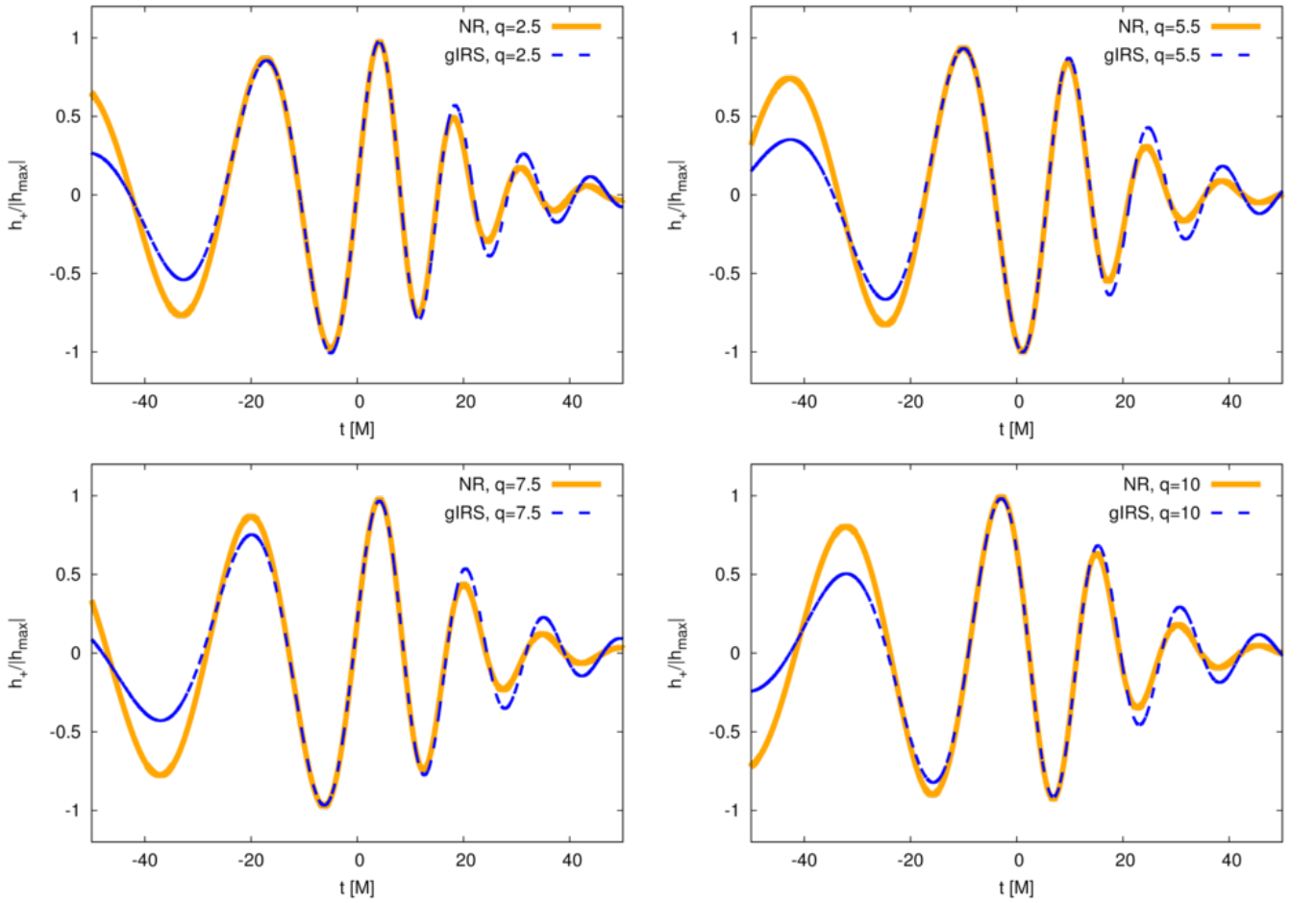


FIG. 4. The panels present the merger model introduced in the main text. We present a direct comparison with waveforms used to calibrate the model, i.e., those with mass-ratio $q = \{2.5, 10\}$, and two additional cases to exhibit the performance of this approach. For reference, $h_+ = \Re[h_{\text{merger}}]$, where h_{merger} is given by Eq. (30) in the main text.

used in the calibration of the free parameters b , c and κ , i.e., numerical simulations for compact binary systems with $q = \{3.5, 5.5, 7.0, 7.5, 9.5\}$.

Finally, we obtain the merger waveform by

$$h^{\text{merger}}(t) = h_+^{\text{merger}} - i h_\times^{\text{merger}} = A(t) e^{-i\Phi_{\text{gIRS}}(t)}, \quad (30)$$

$$\Phi_{\text{gIRS}}(t) = \int_{t_0}^t \omega(t) dt, \quad (31)$$

where $\omega(t)$, $A(t)$ are given by Eq. (19), (21), respectively, and t_0 is a fiducial value within the range of applicability of the gIRS model. Figure 4 shows the regime of applicability of the gIRS model for a variety of compact binary systems, including waveforms that were used for its calibration, and NR simulations that we only use to test the reliability of this scheme.

To combine the inspiral model from Equation (7) with the gIRS model given by Equation (30), we proceed as follows:

- For the inspiral evolution, $h^{\text{inspiral}}(t)$, we define $t = 0$ at 15Hz.
- In the merger waveform, we introduce the free parameters Δt and Φ_0 in Equations (30) and (31), i.e., $t \rightarrow t + \Delta t$ and $\Phi_{\text{gIRS}} \rightarrow \Phi_{\text{gIRS}} + \Phi_0$.
- To compute Δt and Φ_0 , we construct a polynomial using the last three data samples of the inspiral waveform prior to the merger attachment, and require that at the attachment time t^* :
 - * the inspiral and merger waveform are continuous: $h^{\text{inspiral}}(t^*) = h^{\text{merger}}(t^*)$,
 - * the inspiral and merger waveform are differentiable: $\dot{h}^{\text{inspiral}}(t^*) = \dot{h}^{\text{merger}}(t^*)$.
- To find the optimal value of attachment, t_{opt}^* , we repeat the procedure described in the previous item for different t^* , and use the t^* that maximizes the overlap between the ax -waveform in the zero eccentricity limit and its SEOBNRv2 counterpart.

The full IMR waveform is then written as follows:

$$h(t) = \begin{cases} h^{\text{inspiral}}(t) & t \leq t_{\text{opt}}^*, \\ h^{\text{merger}}(t + \Delta t, \Phi_{\text{gIRS}} + \Phi_0) & t \geq t_{\text{opt}}^*. \end{cases} \quad (32)$$

Figure 5 shows two sample waveforms. The top panel shows a BBH system with component masses $(10M_\odot, 10M_\odot)$. The total mass of this system is such that it merges at a time when the system has undergone circularization due to GW emission. The bottom left inset in this panel shows the signatures of eccentricity at low frequency, whereas the top right inset shows that the system has undergone circularization prior to merger. The bottom panel shows a BBH system with component masses $(37.5M_\odot, 7.5M_\odot)$. Since this system is heavier than the previous one, it merges at lower frequencies but still circularizes before merger.

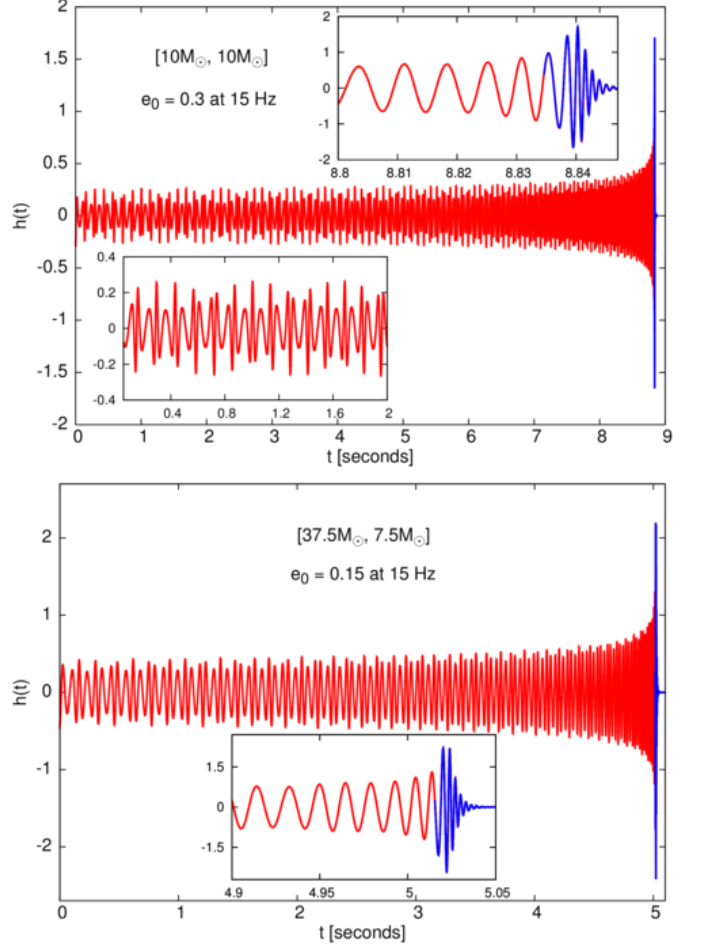


FIG. 5. The panels present the inspiral (red) and merger-ringdown (blue) evolution of two binary black hole systems (see Eq. (32)). The top panel presents a BBH system with component masses $(10M_\odot, 10M_\odot)$ with an initial orbital eccentricity $e_0 = 0.3$ at a GW frequency $f_{\text{GW}} = 15\text{Hz}$. This panel has two insets that show the imprint of eccentricity at low frequencies, and the late-time evolution of this system when the eccentricity has been radiated away. Bottom: BBH system with mass-ratio $q = 5$, total mass $M = 45M_\odot$ and $e_0 = 0.15$ at $f_{\text{GW}} = 15\text{Hz}$.

F. Computational Cost

Another important aspect of the ax -model is its computational efficiency. We have benchmarked the performance of the code introduced in this article using the Campus Cluster of the University of Illinois at Urbana-Champaign (CCUIUC). The specifications of the processors used to carry out this work are: Intel(R) Xeon(R) CPUs E5-2660 at 2.20 GHz.

In order to take into account the fluctuation in performance of compute nodes at the CCUIUC, we compute a waveform for a given set of parameters fifteen times, and quote the average time in Figure 6. Assuming an initial frequency of 15Hz, Figure 6 indicates that the time taken by our code to generate a waveform for binaries with to-

tal mass $M = 10M_\odot$ and $e_0 = 0.4$ is about 0.5 seconds, and ten times faster for quasi-circular systems. Reducing the starting evolution frequency to 10Hz increases the computational cost by about a factor of two.

These results indicate that the ax -waveform model is fast enough to perform large scale parameter estimation studies over the BBH parameter space that can be detected with aLIGO. We are currently implementing this code in the LIGO Algorithms Library [82].

G. Data analysis toolkit

In order to quantify the performance of the ax -model in the zero eccentricity limit, we introduce basic GW data analysis tools. Given two signals h and s , the noise-weighted inner product is defined as

$$(h|s) = 2 \int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}^*(f)\tilde{s}(f) + \tilde{h}(f)\tilde{s}^*(f)}{S_n(f)} df, \quad (33)$$

where $S_n(f)$ represents the power spectral density (PSD) of the detector noise, and $\tilde{h}(f)$ is the Fourier transform of $h(t)$. We take the lower limit of the integral to be $f_{\min} = 15$ Hz, and $f_{\max} = 4096$ Hz. We generate the waveforms using a sample rate of 8192 Hz. The matched-filter signal-to-noise ratio (SNR) is given by

$$\rho = \frac{(s|h)}{\sqrt{(h|h)}}. \quad (34)$$

Using Eq. (33) we construct the normalized waveform:

$$\hat{h} = h (h|h)^{-1/2}, \quad (35)$$

and the normalized overlap

$$\mathcal{O}(h, s) = \max_{t_c, \phi_c} (\hat{h}|\hat{s}_{t_c, \phi_c}), \quad (36)$$

where \hat{s}_{t_c, ϕ_c} indicates that the normalized waveform \hat{s} has been time- and phase-shifted. The Fitting Factor (\mathcal{FF}) is defined as the maximum value of maximized normalized overlaps between a GW signal h^e and all members h_b^T of a bank of template waveforms [83]

$$\mathcal{FF} = \max_{b \in \text{bank}} \mathcal{O}(h^e, h_b^T). \quad (37)$$

The observed SNR ρ' is related to the optimal SNR ρ and the fitting factor through the relation:

$$\rho' = \mathcal{FF}\rho. \quad (38)$$

The waveforms detected by the aLIGO detectors are a combination of the two independent GW polarizations $h_+(t)$ and $h_\times(t)$ through the relation [6]:

$$H(t) = F_+(\theta, \varphi, \psi) h_+(t) + F_\times(\theta, \varphi, \psi) h_\times(t), \quad (39)$$

$$F_+(\theta, \varphi, \psi) = -\frac{1}{2} (1 + \cos^2 \theta) \cos 2\varphi \cos 2\psi - \cos \theta \sin 2\varphi \sin 2\psi, \quad (40)$$

$$F_\times(\theta, \varphi, \psi) = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\varphi \sin 2\psi - \cos \theta \cos 2\varphi \cos 2\psi, \quad (41)$$

where (θ, φ) represent the Euler angles of the detector, and ψ is the Euler angle of the polarization plane.

H. Behavior in zero eccentricity limit

In order to exhibit that the ax -model renders the expected evolution for inspiral dominated systems in the quasi-circular limit, in Figure 7 we present the results of overlap calculations between the ax -model and **TaylorT4** at 3.5PN order. Please note that we have used Eq. (16) for this study. Comparisons with **TaylorT4** at 2PN, 2.5PN and 3PN render a similar behavior, and have the correct asymptotic behavior in the zero eccentricity limit. In these calculations we assume that the binaries are optimally oriented, i.e., $F_+ = 1, F_\times = 0$. We use the Zero Detuned High Power sensitivity configuration for Advanced LIGO [84] and a low frequency cut-off of 15 Hz.

We have also explored the performance of the ax -model in the quasi-circular limit using non-spinning IMR SEOBNRv2 waveforms. For this study we have used the improved inspiral evolution of the ax model given by Eq. (18). We consider a BBH population with component masses $m_{1,2} \in [5M_\odot, 75M_\odot]$, i.e., mass-ratios up to $q = 15$. In Figure 8 we present the overlap between the IMR ax -model with $e = 0$ and SEOBNRv2 for two different scenarios of aLIGO sensitivity [9]. The left panel corresponds to the Zero Detuned High Power sensitivity configuration for aLIGO, using a lower frequency cut-off of 15Hz. The right panel represents the ‘mid aLIGO’ sensitivity configuration, which serves as proxy for the upcoming observing runs O2/O3 [9], using a low frequency cut-off of 25Hz.

In the left panel of Figure 8 we find overlaps $\mathcal{O} \gtrsim 0.95$, indicating that the quasi-circular limit of the IMR ax -model can reproduce the dynamical evolution predicted by the SEOBNRv2 model over a wide region of the BBH parameter space. We should take these results, even if they are positive, with a grain of salt since neither of these models have been calibrated with NR simulations that represent systems with $q > 10$. The right panel of Figure 8 indicates that agreement between the IMR ax -model in the zero eccentricity limit and SEOBNRv2 is better when we consider the ‘mid aLIGO’ sensitivity configuration, which is expected given its narrower sensitive frequency band.

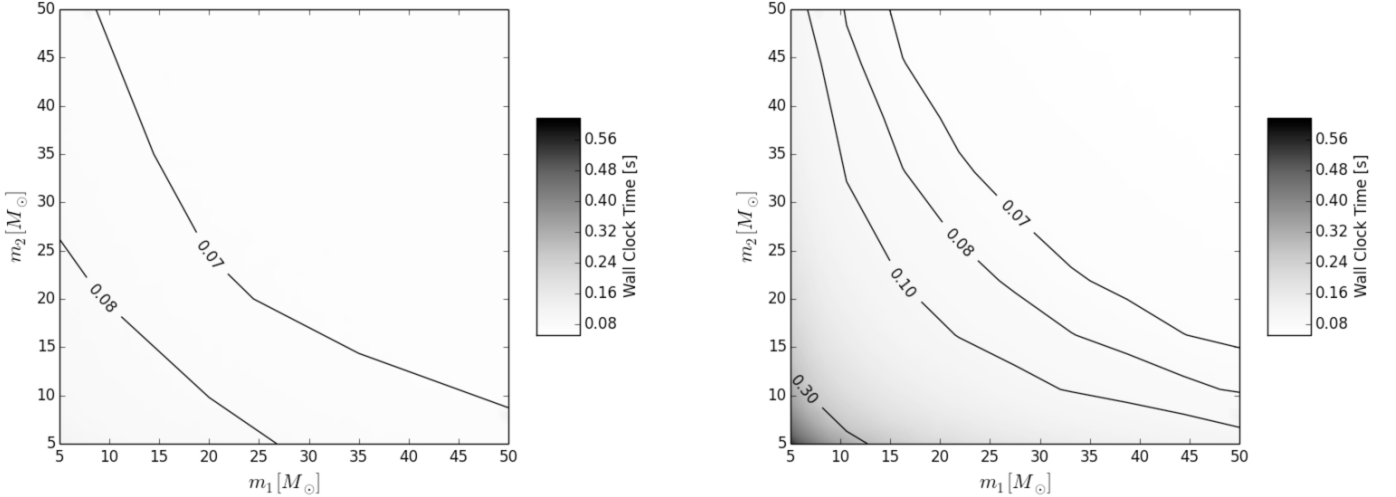


FIG. 6. The panels present the time our code takes to generate a waveform, averaged over 15 iterations, for a given set of parameters. We assume that the binary systems have an initial eccentricity e_0 at a gravitational wave frequency of 15Hz. Left panel: $e_0 = 0$. The contour lines indicate how fast we can generate IMR waveforms in different regions of the BBH parameter space under consideration. Right panel: as left panel but for systems with $e_0 = 0.4$.

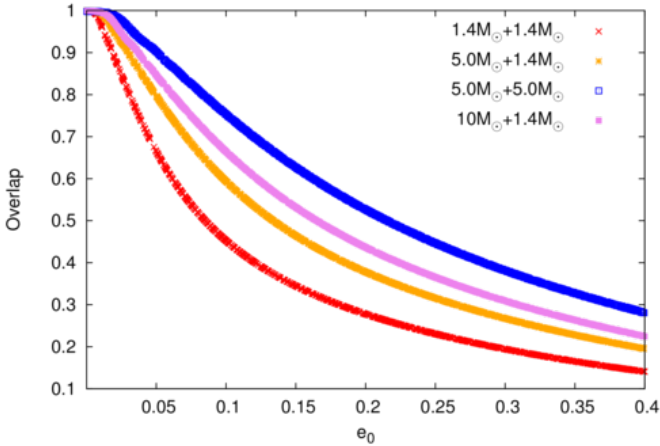


FIG. 7. Overlap between the ax -model and the approximant TaylorT4 including 3.5PN corrections. We have used the Zero Detuned High Power sensitivity configuration for Advanced LIGO and a lower frequency cut-off of 15Hz.

Figure 8 is the first comparison of an IMR eccentric waveform with a state-of-the-art quasi-circular IMR waveform model such as SEOBNRv2. The panels in Figure 8 indicate that our IMR ax model can reproduce non-spinning SEOBNRv2's dynamics with an average overlap $\mathcal{O}_{\text{average}} \sim 0.95$ and that some regions of parameter space have $\mathcal{O}_{\text{max}} \sim 0.99$. *This is the first IMR eccentric model in the literature that has this level of agreement with SEOBNRv2 for BBH systems with mass-ratios $1 \leq q \leq 15$.* We notice, however, that the model has anomalously low overlaps, $\mathcal{O}_{\text{min}} \sim 0.9$, for binaries in a narrow band of mass-ratios centered at $q \sim 4$. We can un-

derstand this undesirable feature by taking a closer look at the construction of our waveform model. In Figure 5 we see that we combine a PN-based eccentric inspiral model with a merger waveform very late in the inspiral evolution. This late-time attachment, however, does not work uniformly well in the binary parameter space, and introduces anomalous features in the model for $q \sim 4$. While the gIRS model provides a good description of the merger dynamics in the vicinity of the light-ring, its accuracy deteriorates rapidly several cycles before merger, cf. Figure 4. In different words, we are pushing the PN equations of motion to the limit of their applicability to ensure we get the best possible overlap with SEOBNRv2. We comment in Section IV on possible improvements to the gIRS model.

Figure 8 demonstrates the importance of the amended inspiral dynamics in Eq. (18). Without those amendments, typical overlap values between the ax model and SEOBNRv2 are $\mathcal{O} \lesssim 0.5$ for systems with mass-ratios $q \gtrsim 4$. Thus, the corrections that have to be implemented to ensure that the minimum overlap between the IMR ax -model and SEOBNRv2 satisfies $\mathcal{O} \gtrsim 0.99$ over the whole BBH space are within reach with additional work that we describe in Section IV. At present, the ax -model presented in this article can be used to: (i) explore how well eccentricity can be measured in parameter estimation studies. We can do this by injecting ax signals in real data and do a parameter estimation analysis with ax templates; (ii) study the bias incurred in parameter estimation studies caused by the intrinsic inaccuracies of the ax -model. We can do this by injecting NR waveforms and doing a parameter estimation study with ax templates; (iii) furthermore, we can study how well eccentric BBH signals can be recovered with non-eccentric

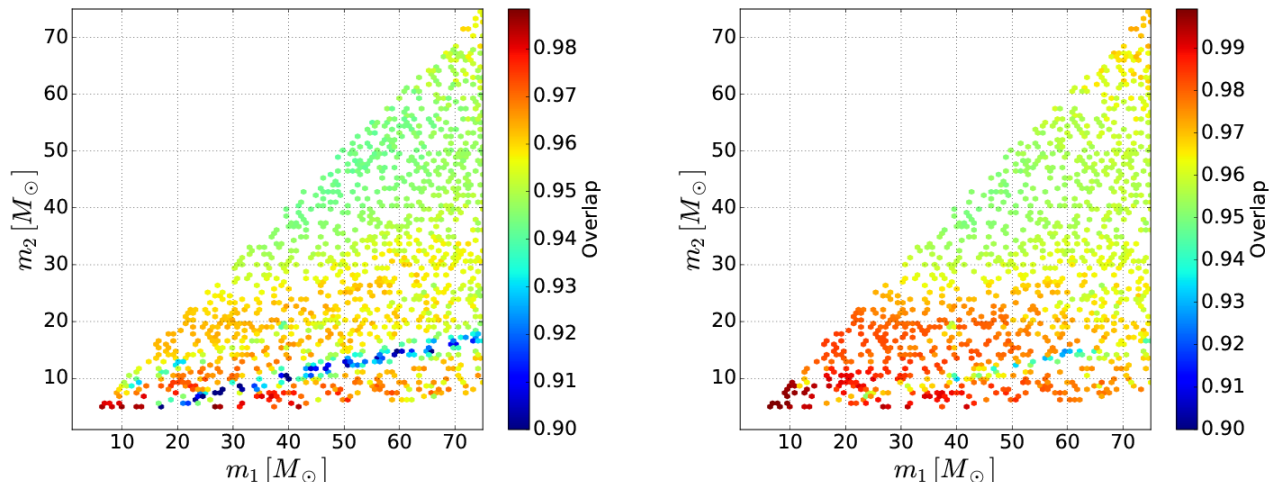


FIG. 8. Overlap between the ax -model in the zero eccentricity limit and the SEOBNRv2 model. Left panel: the overlap is computed from an initial gravitational wave frequency of 15Hz using the Zero Detuned High Power sensitivity configuration for aLIGO. Right panel: the overlap is computed from an initial gravitational wave frequency of 25 Hz using the mid aLIGO sensitivity curve described in Ref. [9]. We note that in both cases a large portion of the parameter space under consideration is accurately reproduced by the ax -model with $\mathcal{O} \gtrsim 0.95$.

waveform templates by using eccentric ax -waveforms as injections to be recovered with a template bank consisting of non-eccentric waveform templates. This will be the topic of Section III.

III. DETECTABILITY OF ECCENTRIC UNEQUAL MASS BINARIES

A previous study [30] of the importance of eccentricity to model and detect BNSs that have moderate values of residual eccentricity used the x -model of Ref. [42]. Now that we have developed the IMR ax -model, we are equipped to extend that analysis to systems that have asymmetric mass-ratios including the inspiral, merger and ringdown phases. *To the best of our knowledge, this is the first analysis of this nature in the literature.*

The first part of this analysis is related to quantifying the effect of eccentricity in the dynamical evolution of stellar mass BBH and NSBH systems. We carry out this study by directly comparing SEOBNRv2 waveforms against IMR ax waveforms using astrophysically motivated values of eccentricity, i.e., $e_0 \in [0, 0.4]$, where e_0 is defined at $f_{\text{GW}} = 14\text{Hz}$. The results of this study are presented in Figure 9 for compact binaries with mass-ratios $q = \{1, 3, 5, 7\}$. We restrict this study to systems with total mass $M \leq 45M_\odot$, since such binaries will effectively circularize by the time they reach their ISCO, i.e., $e_{\text{ISCO}} \leq 0.05$ — see Figure 1. These results were obtained using $f_{\text{min}} = 15\text{Hz}$ (see Eq. (33)), and the Zero Detuned High Power PSD of aLIGO.

The results presented in Figure 9 indicate that low mass binaries with very asymmetric mass-ratios are the sys-

tems that differ the most from their quasi-circular counterparts. For instance, the overlaps between SEOBNRv2 and IMR ax -waveforms for a $(5M_\odot, 5M_\odot)$ BBH and a $(8.75M_\odot, 1.25M_\odot)$ NSBH binary that enter the aLIGO band with $e_0 = 0.1$ at $f_{\text{GW}} = 14\text{Hz}$ are: $\mathcal{O} \sim 0.75$ and $\mathcal{O} \sim 0.6$, respectively. This significant drop in overlap is caused by several factors: (i) eccentricity corrections have a cumulative effect in the orbital phase of waveform signals. Therefore, the orbital phase of long lived eccentric signals will significantly deviate from their quasi-circular counterparts. In a population of binaries with total mass M , those with the most asymmetric mass-ratios have the longest lifespan. Therefore, we expect that the most significant drop in overlap between eccentric and quasi-circular systems should correspond to NSBHs and BBHs with asymmetric mass-ratios, as shown in Figure 9; (ii) eccentricity reduces the lifespan of waveform signals. Signals with $e_0 = 0.4$ are a factor ~ 2 shorter than their quasi-circular counterparts. Therefore, it is no surprise that the overlap between these type of signals and SEOBNRv2 is $\mathcal{O} \sim 0.2$. Putting (i) and (ii) together, we can understand that this effect is exacerbated for low mass, asymmetric mass-ratio systems. On the other hand, more massive systems spend less time in the aLIGO band, preventing eccentricity corrections to accumulate. As a result, the overlap between quasi-circular templates and eccentric binaries with $M \sim 45M_\odot$ and $e_0 \leq 0.1$ is $\mathcal{O} \geq 0.9$.

Having developed a basic understanding on the effect of eccentricity in terms of the total mass and mass-ratio of compact binaries, we now turn our attention to the detectability of eccentric signals using template banks of quasi-circular waveforms. We can quantify the recov-

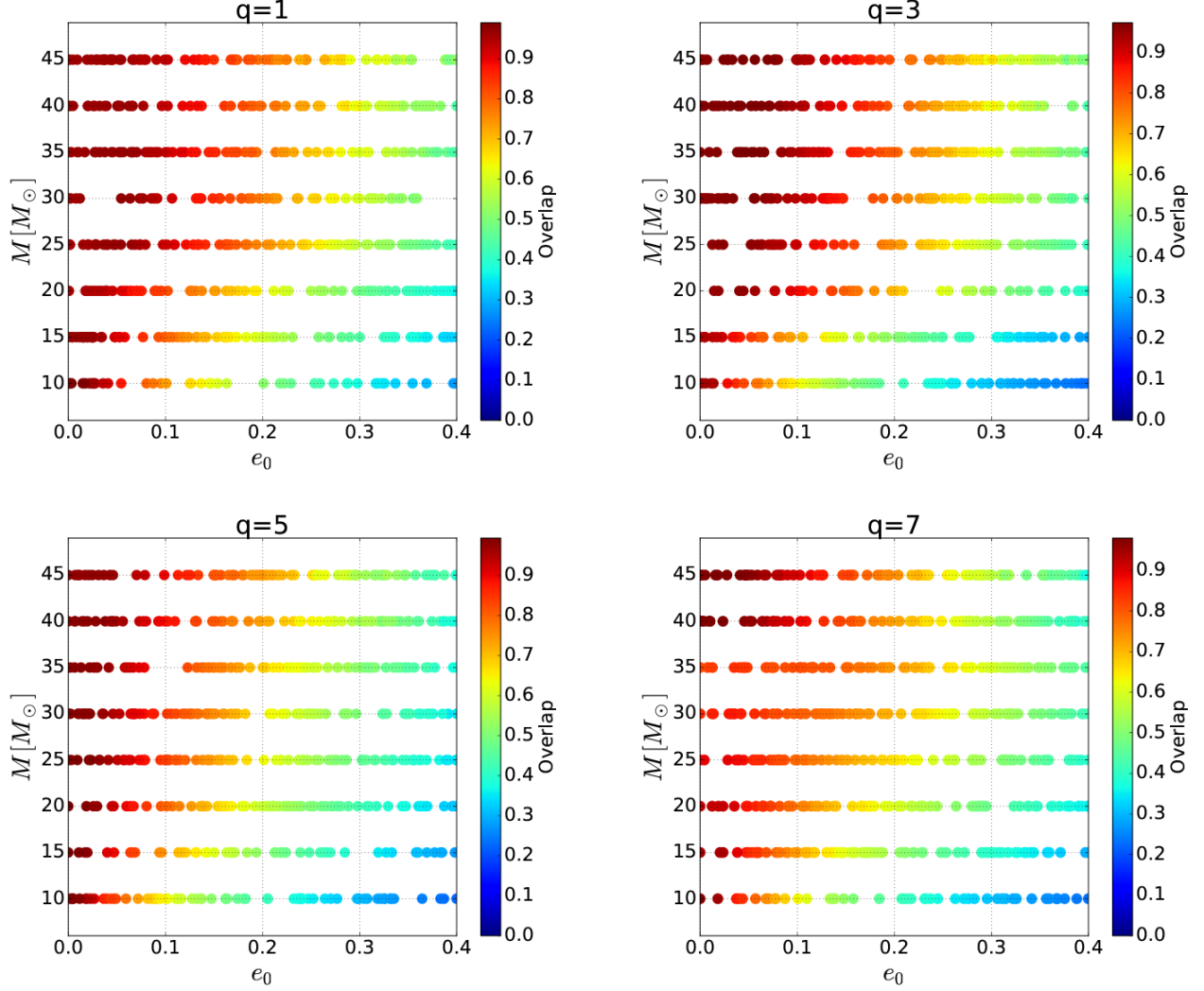


FIG. 9. The panels present overlap calculations between IMR ax and SEOBNRv2 waveforms. The IMR ax waveforms are generated for binaries that enter aLIGO band with eccentricities $e_0 \in [0, 0.4]$ at $f_{\text{GW}} = 14\text{Hz}$. The overlaps are computed from an initial gravitational wave frequency $f_{\text{min}} = 15\text{Hz}$ (see Eqs. (33)) using the Zero Detuned High Power sensitivity configuration for aLIGO.

ery of non-spinning, eccentric signals using two types of template banks of quasi-circular waveforms: (i) SEOBNRv2 template banks allow us to do recovery with non-spinning and aligned-spin templates. Therefore, we can test whether aligned-spin templates do capture the effect of eccentricity. This is important, because GW searches with aLIGO utilize aligned-spin templates, so this is a relevant question when assessing aLIGO’s sensitivity to eccentric systems. Unfortunately, this yields ambiguities at small e_0 , because small e_0 ax -injections do not perfectly agree with SEOBNRv2. This information is conveyed in Figure 9: overlaps with $e_0 \lesssim 0.05$ are comparable to their quasi-circular counterparts, i.e., it is not possible to make clear cut statements about the effect of eccentricity for systems with $e_0 \lesssim 0.05$. Rather, these overlap calculations provide information about the accuracy of

the ax -waveforms in the zero eccentricity limit. On the other hand, overlaps for systems with $e_0 \gtrsim 0.1$ significantly drop from the quasi-circular case, which indicates that for $e_0 \gtrsim 0.1$ we are probing predominantly the effect of eccentricity. As discussed above, these boundaries depend on the total mass and mass-ratio of the systems, with high masses being less sensitive to eccentricity. (ii) Conversely, with ax -template banks we can make rigorous statements about recovery efficiency of small eccentricity injections with *non-spinning* templates, but cannot make statements about recovery with aligned-spin templates. For the present study, we choose the first approach and consider two scenarios: (a) we set the spin of the binary components to zero and construct a template bank that describes binaries with non-spinning components on quasi-circular orbits, (b) we construct a tem-

plate bank that describes binaries on quasi-circular orbits whose components have spin in the z direction only. To quantify the effectualness with which these template banks recover eccentric signals, we computed \mathcal{FF} 's from an initial $f_{\min} = 15\text{Hz}$ using the Zero Detuned High Power sensitivity configuration for aLIGO. The simulated eccentric signals enter the aLIGO band with initial eccentricity e_0 at $f_{\text{GW}} = 14\text{Hz}$.

In order to ensure that the template bank discreteness does not affect the recovery of simulated eccentric signals, we constructed template banks with 5×10^5 , 10^6 and 1.5×10^6 waveforms, and tested the convergence of the \mathcal{FF} 's presented below. We found that the bank constructed with non-spinning waveforms is a proper subset of the spin-aligned bank when we densely sample the parameter space using 1M waveforms. We compared the \mathcal{FF} 's obtained using the spin-aligned bank with 10^6 and 1.5×10^6 waveforms and confirmed that the \mathcal{FF} 's were exactly the same. In different words, this consistency check indicates that the results we present below represent the true maximum \mathcal{FF} 's, which surpass the effect of template bank discreteness. In Figure 10, we show the coverage of the mass parameter space (m_1, m_2) used for the construction of these template banks, and the 8×10^3 eccentric simulated signals or ‘injections’.

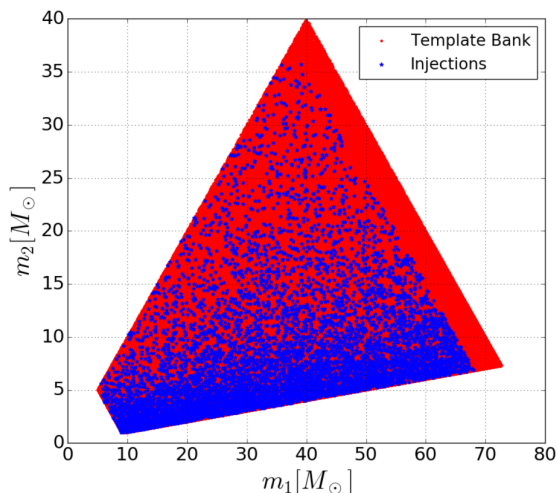


FIG. 10. The panel shows the coverage of the mass parameter space (m_1, m_2) using 10^6 quasi-circular template waveforms. We also show the mass distribution of the 8×10^3 simulated eccentric signals or ‘injections’.

In Figure 11 we investigate recovery of ax -waveforms with $e_0 \leq 0.05$ using non-spinning and spin-aligned SEOBNRv2 banks. Figure 11 indicates that template bank maximization does increase the overlap results presented in Figure 9. Furthermore, template banks of spin-aligned SEOBNRv2 waveforms recover non-spinning, *mildly* eccentric ax -waveforms with higher \mathcal{FF} 's than their quasi-circular, non-spinning counterparts. This is because the additional degrees of freedom of spin-aligned

waveforms can be optimally combined to reproduce the dynamical evolution of non-spinning, weakly eccentric ax -waveforms. The panels in this Figure include a black and a green star, which represent the GW transients detected by aLIGO: GW150914 with $M^* = 67M_\odot$, and GW151226 with $M^* = 22M_\odot$, respectively. Our results show that a template bank of spin-aligned SEOBNRv2 waveforms can recover GW150914 with $\mathcal{FF} \geq 0.98$ and GW151226 with $\mathcal{FF} \geq 0.97$ if $e_0 \leq 0.05$. As discussed before, we should take these results with a grain of salt because in this low eccentricity regime \mathcal{FF} 's may be dominated by the modeling errors of ax -waveforms in the zero eccentricity limit.

Let us now consider astrophysically realistic eccentricities, $e_0 = 0.1$ and $e_0 = 0.15$. At these eccentricities, ax vs SEOBNRv2 overlaps have already significantly deteriorated relative to the $e_0 = 0$ comparison (cf. Figure 9), so we expect that we are really probing the effect of eccentricity in our comparisons. We notice that template bank maximization, given by the \mathcal{FF} results in Figure 12, does not significantly improve the overlap calculations presented in Figure 9. This suggests that the manifold generated by the eccentric signals is orthogonal to the usual quasi-circular manifold. Furthermore, recovery with the spin-aligned SEOBNRv2 template bank does not render significantly better results than its non-spinning counterpart. This implies that the spin-aligned degrees of freedom of the template bank are orthogonal to the eccentric degree of freedom of the injection manifold. Regarding the recovery of GW150914 and GW151226, we notice that these transients can be recovered with spin-aligned SEOBNRv2 templates with $\mathcal{FF} \geq 0.95$ if $e_0 \leq 0.15$ and $\mathcal{FF} \geq 0.94$ if $e_0 \leq 0.1$, respectively. Furthermore, the impact of total mass and mass-ratio in the recovery of eccentric signals is significant in this regime. For $e_0 = 0.1$, an equal mass $10M_\odot$ BBH and a $10M_\odot$ NSBH with $q = 8$ are recovered with $\mathcal{FF} = 0.90$ and $\mathcal{FF} = 0.86$, respectively. These results indicate that BBH and NSBH systems with astrophysically motivated values of eccentricity ($e_0 \sim 0.1$) will not be recovered with matched-filtering algorithms based on quasi-circular waveforms [7]. In general, we find that systems with $e_0 \geq 0.15$ are poorly recovered, $\mathcal{FF} \leq 0.93$.

For completeness, let us finally investigate large eccentricities, $e_0 = 0.2$ and $e_0 = 0.3$, which — according to present astrophysical understanding — are hard to achieve [7, 10, 17]. Nevertheless, it is important to know how sensitive aLIGO is to such eccentric binaries, to independently verify astrophysical theory. As in the case of astrophysically motivated values of eccentricity (cf. Figure 12), Figure 13 indicates that the eccentric signal manifold is orthogonal to the non-spinning and spin-aligned template bank manifolds. Furthermore, since the recovery with both types of SEOBNRv2 banks is similar, we infer that the eccentricity degree of freedom of the signal manifold cannot be captured with the additional degrees of freedom of the spin-aligned SEOBNRv2 bank. These results also indicate that it will be unfeasible for

quasi-circular searches to capture GW signals with eccentricities $e_0 \geq 0.2$. For $e_0 = 0.2$, an equal mass $10M_\odot$ BBH and a $10M_\odot$ NSBH with $q = 8$ have $\mathcal{FF} = 0.81$ and $\mathcal{FF} = 0.73$, respectively. Recovery deteriorates very significantly for $e_0 \geq 0.3$ — most eccentric signals are recovered with $\mathcal{FF} \leq 0.8$, and typical NSBH systems have $\mathcal{FF} \leq 0.6$.

Up to this point we have discussed recovery of non-spinning template banks and aligned-spin template banks in parallel. We now investigate in further detail a different aspect of the impact of aligned-spin SEOBNRv2 template banks. To do so we compute the effective spin, χ_{eff} , of the spin-aligned template waveforms that best recovered eccentric signals in our simulations, i.e.,

$$\chi_{\text{eff}} = \frac{m_1^t}{M_t} \chi_1^z + \frac{m_2^t}{M_t} \chi_2^z - \frac{38\eta^t}{113} (\chi_1^z + \chi_2^z), \quad (42)$$

where (m_1^t, m_2^t) are the template masses, (χ_1^z, χ_2^z) are the dimensionless spins of the templates, and $M_t = m_1^t + m_2^t$, $\eta^t = m_1^t m_2^t / M_t^2$. Figure 14 presents the χ_{eff} for eccentric compact binary populations with $e_0 \leq 0.3$. These results present the following global picture: template bank optimization with spin-aligned templates improves overlaps for eccentric populations with $e_0 \leq 0.05$, and slightly increases recovery with respect to the non-spinning SEOBNRv2 bank. This is because the spin-aligned degrees of freedom: (i) compensate for the modeling errors of ax -waveforms in the $e_0 \rightarrow 0$ limit; and (ii) are able to reproduce the minor shortening effect of weakly eccentric signals. However, for $e_0 \geq 0.1$ the modeling errors of quasi-circular ax -waveforms are small compared to the effect of eccentricity, and recovery is dominated by eccentricity. We notice that in this eccentricity regime, χ_{eff} only achieves significant values for low total mass systems, i.e., spin does not play a significant role in eccentric signal recovery. As we discussed above, this implies that non-spinning, eccentric populations with $e_0 \geq 0.1$ define a manifold that is predominantly orthogonal to the quasi-circular, non-spinning and spin-aligned manifolds.

The results presented in this Section clearly indicate that matched-filtering algorithms tuned for quasi-circular waveforms will not be effectual at recovering stellar mass BBH and NSBH systems with astrophysically motivated values of eccentricity, i.e., $e_0 \sim 0.1$ [7]. We have also shown that the two GW transients already detected by the aLIGO detectors could have had non-negligible values of residual eccentricity at $f_{\text{GW}} = 14\text{Hz}$, and still be detected with high \mathcal{FF} values using spin-aligned SEOBNRv2 template banks. These results are the first in their kind in the literature.

IV. CONCLUSION

We have developed a waveform model for eccentric compact binaries that represents the inspiral, merger and ringdown, and that reproduces zero eccentricity binary

waveforms much more accurately than previous eccentric waveform models. With this model we study the importance of including eccentricity in detecting eccentric NSBH and BBH systems with aLIGO. We show that using the design sensitivity of aLIGO and a lower frequency cut-off of 15Hz, the IMR ax -model can reproduce the SEOBNRv2 model in the zero eccentricity limit with overlap values $\mathcal{O} \gtrsim 0.95$ over a wide range of the stellar mass BBH and NSBH parameter space that is accessible to aLIGO.

Using our IMR ax model we explore the detectability of eccentric compact binaries. Our results indicate that template banks of quasi-circular, spin-aligned SEOBNRv2 waveforms can recover GW150914 with $\mathcal{FF} \geq 0.95$ if $e_0 \leq 0.15$, and GW151226 with $\mathcal{FF} \geq 0.94$ if $e_0 \leq 0.1$. We have also found that template banks of quasi-circular, spin-aligned waveforms can improve the recovery of low total mass moderately eccentric signals. Our results also indicate that low mass BBH and NSBH systems with astrophysically motivated values of eccentricity ($e_0 \sim 0.1$) will be poorly recovered with available quasi-circular matched-filtering algorithms ($\mathcal{FF} \leq 0.85$). In order to detect these events, it is necessary to develop new data analysis algorithms that specifically target eccentric GW sources.

A key assumption in the construction of our ax -model is that compact binaries circularize prior to merger. We explore the validity of this assumption and find that we can cover a large portion of the parameter space of compact binaries that aLIGO will be able to detect. In order to minimize the effect of inherent waveform inaccuracies in the ax -model, particularly in the context of parameter estimation studies, we are exploring two ways to enhance its accuracy in the $e \rightarrow 0$ limit. The first improvement deals with the hybridization between inspiral-PN model and gIRS merger-ringdown model: in its current version the ax -model consists of a simple hybridization between the PN-inspiral evolution and the gIRS model we have described in Section III E. The key for this procedure to work requires that both frameworks meet at an optimal frequency where they render the correct dynamical evolution. The results we have obtained in this work suggests that using up-to-date results from the self-force formalism and PN theory provides a robust framework to capture the inspiral dynamics of compact binaries with asymmetric mass-ratios. The enhanced inspiral evolution we have constructed is good to *explore* the late time dynamics of BBHs, but it can only go so far. At the other end of the spectrum, the gIRS model is reliable in the vicinity of the light-ring. We can see in Figure 4 that this approach starts to deteriorate when we push the model several cycles prior to the merger event. Therefore, a critical correction to further improve the IMR ax -model is the development of a new merger-ringdown prescription that captures the true dynamical evolution *several* cycles before merger, and which can provide a wider window of frequencies to hybridize the inspiral evolution with the merger phase.

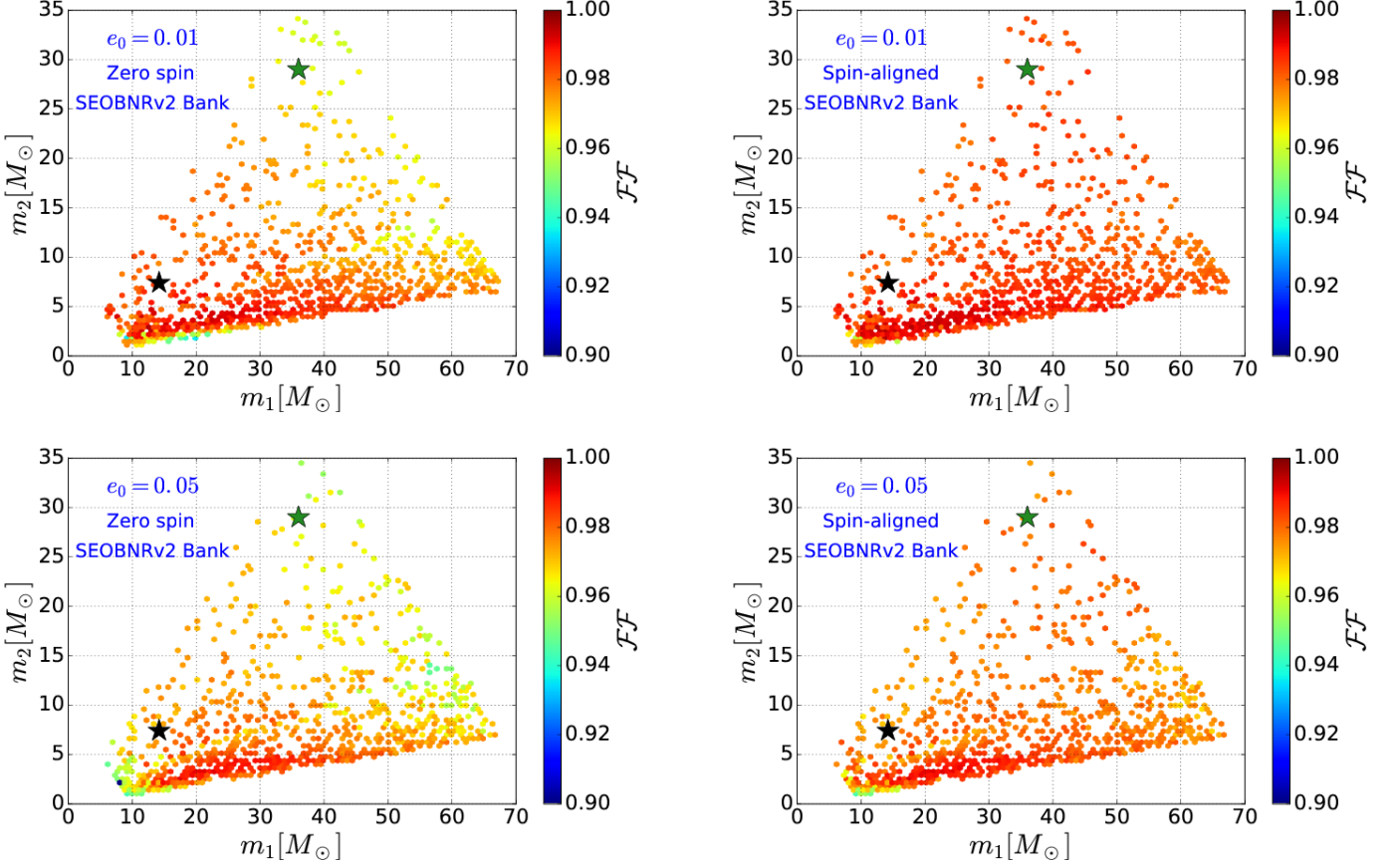


FIG. 11. Left panels: effectiveness of a bank of quasi-circular, non-spinning SEOBNRv2 templates to recover a population of eccentric, non-spinning signals. Right panels: recovery of non-spinning eccentric injections with an *aligned-spin* template bank of SEOBNRv2 waveforms. Each panel indicates the eccentricity e_0 at which these systems enter aLIGO band at $f_{\text{GW}} = 14\text{Hz}$. The Fitting Factors (\mathcal{FF}) are computed using $f_{\text{min}} = 15\text{Hz}$ (see Eqs. (33) and (37)), and the Zero Detuned High Power sensitivity configuration for aLIGO. The green and black stars represent the GW transients GW150914 and GW151226, respectively.

Our second planned improvement concerns the inspiral dynamics itself. Presently, 4-6PN terms in the binding energy of compact binaries $E(x, \eta)^{6\text{PN}}$, cf. Eq. (17), only include first order in symmetric mass-ratio corrections. We will further improve the inspiral dynamics by including terms at *second order* in symmetric-mass ratio. Furthermore, building up on [73, 87], we will amend the energy flux prescription, $\dot{E}(x, \eta)^{6\text{PN}}$, by constraining missing η^2 corrections in the energy flux expression used in Eq. 17.

We expect that combining the aforementioned improvements will provide an enhanced performance of the *ax*-model in the $e_0 \rightarrow 0$ limit so that the overlap with SEOBNRv2 templates satisfies $\mathcal{O} \gtrsim 0.99$ over the stellar mass BBH and NSBH parameter space accessible to aLIGO. The results we present in this article indicate that a consistent combination of higher-order PN calculations, self-force corrections and NR can enable the construction of accurate, computationally inexpensive waveform models that encode the dynamics of compact binary systems across the parameter space accessible to aLIGO–

type detectors. These results further support the importance of deriving second order self-force effects [88–94]. Previous studies have strongly relied on self-force calculations for waveform modeling, source detection and parameter estimation studies, and have exhibited their applicability for extreme and comparable mass-ratio systems [54, 73, 95–104]. Moving forward, it is necessary to develop new waveform models that enable the description of compact binaries whose components have non-zero spin and which evolve on eccentric orbits. This work should be pursued in the near future.

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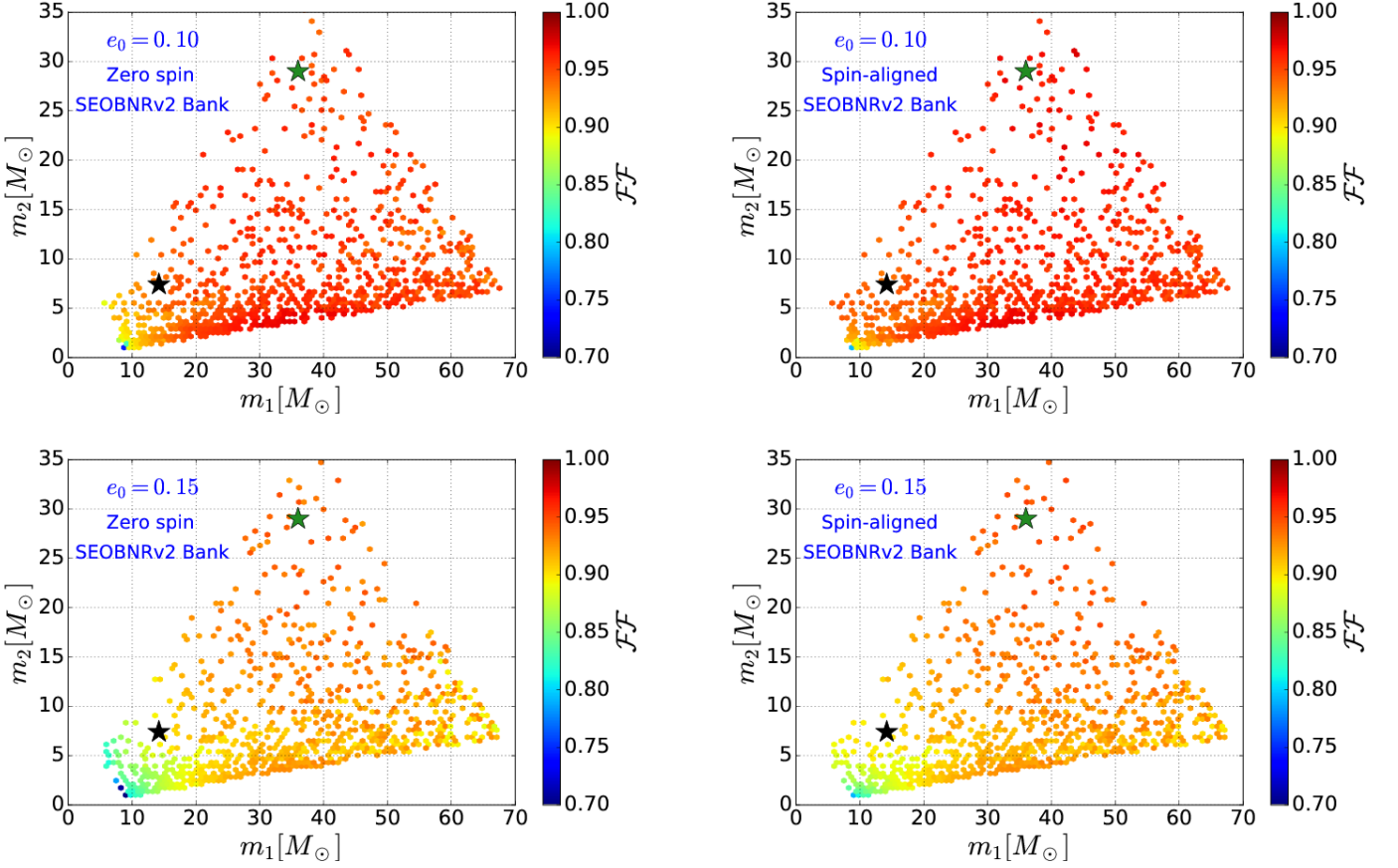


FIG. 12. As Figure 11 but now for compact binary populations with $e_0 = \{0.1, 0.15\}$. Note that the color bar has been adjusted to the range $[0.7, 1]$ to exhibit additional structure for low \mathcal{FF} values.

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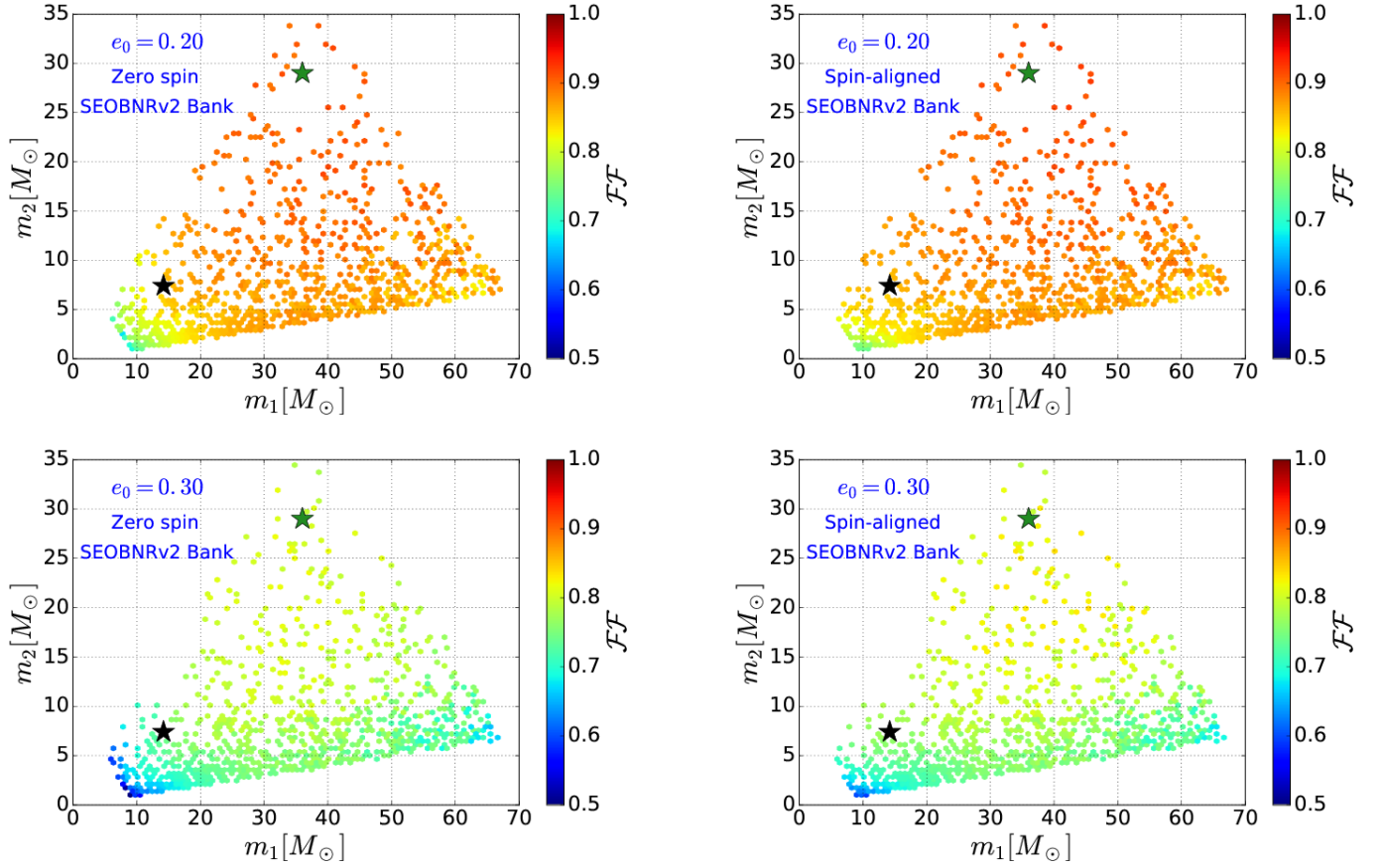


FIG. 13. As Figure 11 but now for compact binary populations with $e_0 = \{0.2, 0.3\}$. Note that the color bar range is $[0.5, 1]$.

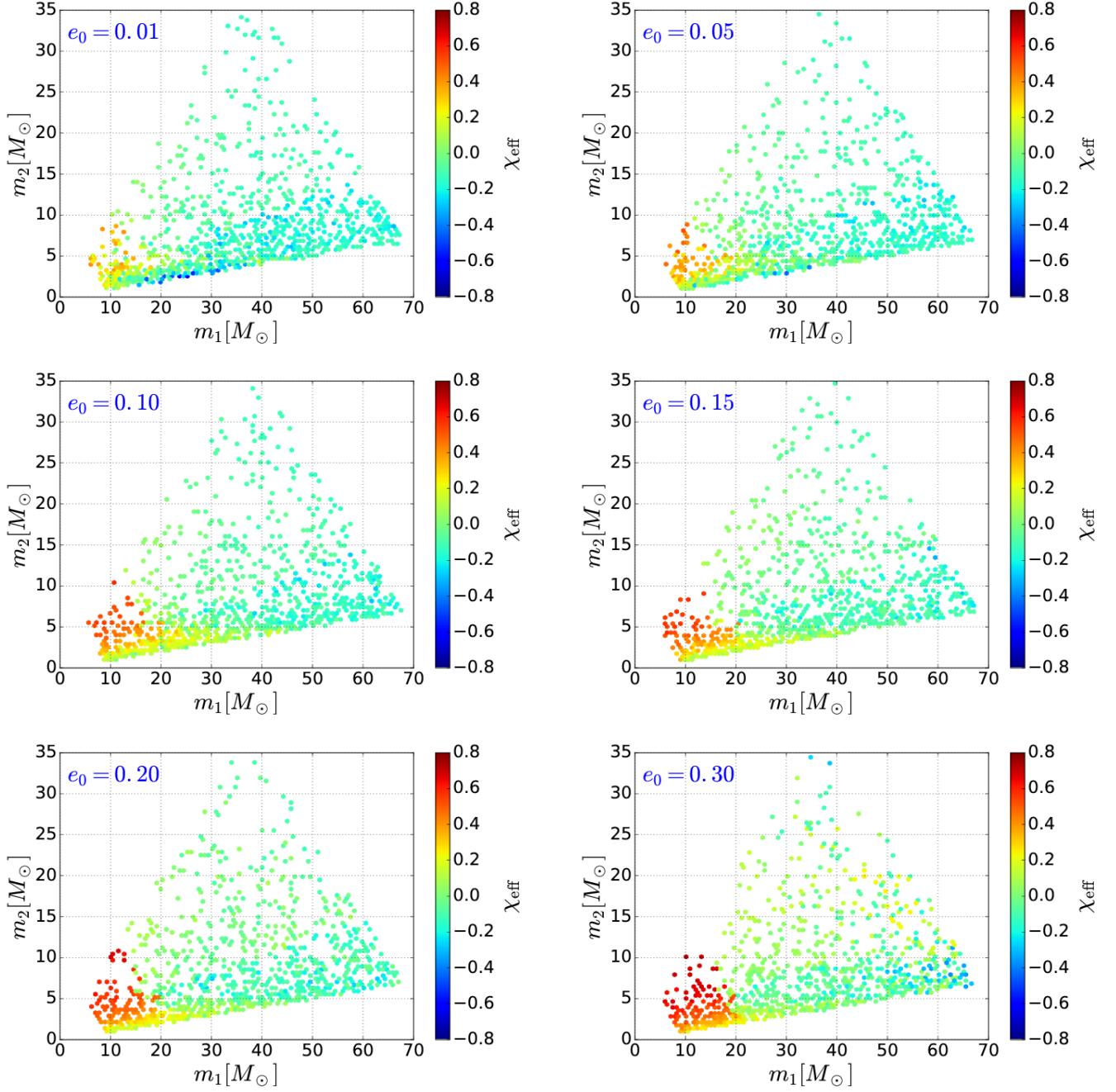


FIG. 14. Effective spin χ_{eff} with which eccentric signals are recovered (see Eq. (42) in the main text). The magnitude of χ_{eff} indicates that spin-aligned SEOBNRv2 template banks significantly improve the recovery of non-spinning, eccentric waveforms for low total mass systems.

Appendix A

Higher-order PN calculations for eccentric binaries have been computed in terms of the mean motion n and e in Ref. [28]. In this Appendix we re-write those results in terms of the gauge-invariant quantity $x = (M\omega)^{2/3}$ and e . To do so we use the following relation between the mean motion n , the gauge-invariant quantity x and e [42, 43]:

$$\begin{aligned}
 Mn = & \frac{x^{3/2}}{(1-e^2)^3} \left[1 - 3e^2 + 3e^4 - e^6 + x(-3 + 6e^2 - 3e^4) + x^2 \left[-\frac{9}{2} + 7\eta + \left(-\frac{33}{4} - \frac{\eta}{2}\right)e^2 + \left(\frac{51}{4} - \frac{13\eta}{2}\right)\eta e^4 \right] \right. \\
 & + x^3 \left[\frac{3}{2} + \eta \left(\frac{457}{4} - \frac{123\pi^2}{32} \right) - 7\eta^2 + \left(-\frac{267}{4} + \eta \left(\frac{279}{2} - \frac{123\pi^2}{128} \right) - 40\eta^2 \right) e^2 + \left(-\frac{39}{2} + \frac{55\eta}{4} - \frac{65\eta^2}{8} \right) e^4 \\
 & \left. \left. + \sqrt{1-e^2}(-15 + 6\eta + (-30 + 12\eta)e^2) \right] \right] + \mathcal{O}(x^{11/2}). \tag{A1}
 \end{aligned}$$

The time evolution of x is given by:

$$M\dot{x} = \dot{x}_{0\text{PN}}x^5 + \dot{x}_{1\text{PN}}x^6 + \dot{x}_{2\text{PN}}x^7 + \dot{x}_{3\text{PN}}x^8 + \dot{x}_{\text{HT}}, \tag{A2}$$

where \dot{x}_{HT} stands for hereditary terms. (x_0, x_1) can be found in [42]:

$$\dot{x}_{0\text{PN}} = \frac{2(37e^4 + 292e^2 + 96)\eta}{15(1-e^2)^{7/2}}, \tag{A3}$$

$$\dot{x}_{1\text{PN}} = \frac{\eta(11717e^6 + 171038e^4 + 87720e^2 - 28(296e^6 + 5061e^4 + 5700e^2 + 528)\eta - 11888)}{420(1-e^2)^{9/2}}, \tag{A4}$$

In this work, we have derived $\dot{x}_{2\text{PN}}$, $\dot{x}_{3\text{PN}}$ and \dot{x}_{HT} :

$$\begin{aligned}
 \dot{x}_{2\text{PN}} = & -\frac{\eta}{45360(1-e^2)^{11/2}} \left[-3e^8 \left(4\eta(163688\eta - 271665) + 1174371 \right) \right. \\
 & + 16e^2 \left(-21\eta \left(-76824\sqrt{1-e^2} + 182387\eta + 46026 \right) - 4033260\sqrt{1-e^2} + 5802910 \right) \\
 & + 32 \left(-9\eta \left(-2016\sqrt{1-e^2} + 6608\eta + 15677 \right) - 45360\sqrt{1-e^2} + 11257 \right) \\
 & + 6e^6 \left(7\eta \left(25200\sqrt{1-e^2} - 1543544\eta + 2931153 \right) - 3 \left(147000\sqrt{1-e^2} + 4634689 \right) \right) \\
 & \left. + 12e^4 \left(\eta \left(2210544\sqrt{1-e^2} - 13875505\eta + 17267022 \right) - 34 \left(162540\sqrt{1-e^2} + 1921 \right) \right) \right], \tag{A5}
 \end{aligned}$$

$$\begin{aligned}
\dot{x}_{\text{3PN}} = & \frac{\eta}{598752000(1-e^2)^{13/2}} \left[25e^{10} \left\{ 2699947161 - 176\eta \left(4\eta(2320640\eta - 2962791) + 16870887 \right) \right\} \right. \\
& + 32e^2 \left\{ 55\eta \left[270(7015568\sqrt{1-e^2} - 9657701)\eta - 8125851600\sqrt{1-e^2} + 38745\pi^2(1121\sqrt{1-e^2} + 1185) \right. \right. \\
& \left. \left. - 901169500\eta^2 + 5387647438 \right] + 31050413856\sqrt{1-e^2} + 358275866598 \right\} \\
& + 128 \left\{ -275\eta \left[81(16073 - 17696\sqrt{1-e^2})\eta \right. \right. \\
& \left. \left. - 1066392\sqrt{1-e^2} + 46494\pi^2(\sqrt{1-e^2} - 45) + 470820\eta^2 + 57265081 \right] - 3950984268\sqrt{1-e^2} + 12902173599 \right\} \\
& + e^8 \left\{ 162(1240866000\sqrt{1-e^2} + 19698134267) - 1100\eta \left[16\eta(-3582684\sqrt{1-e^2} + 137570300\eta - 286933509) \right. \right. \\
& \left. \left. + 27(6843728\sqrt{1-e^2} + 255717\pi^2 + 173696120) \right] \right\} \\
& + 12e^6 \left\{ 55\eta \left[90(52007648\sqrt{1-e^2} + 311841025)\eta \right. \right. \\
& \left. \left. + 3(4305\pi^2(14\sqrt{1-e^2} - 19113) - 5464335200\sqrt{1-e^2} + 767166806) - 17925404000\eta^2 \right] \right. \\
& \left. + 742016570592\sqrt{1-e^2} + 6005081022 \right\} \\
& + 8e^4 \left\{ 55\eta \left[270(71069152\sqrt{1-e^2} + 6532945)\eta \right. \right. \\
& \left. \left. - 74508169680\sqrt{1-e^2} + 116235\pi^2(1510\sqrt{1-e^2} - 4807) - 23638717900\eta^2 + 88628306866 \right] \right. \\
& \left. + 6(332891836596\sqrt{1-e^2} + 8654689873) \right\} \\
& \left. + 40677120(891e^8 + 28016e^6 + 82736e^4 + 43520e^2 + 3072) \log \left(\frac{x(1+\sqrt{1-e^2})}{2(1-e^2)} \right) \right], \tag{A6}
\end{aligned}$$

$$\begin{aligned}
\dot{x}_{\text{HT}} = & \eta x^{13/2} \left[\frac{256\pi}{5} \phi(e) + \left(\frac{256\pi}{1-e^2} \phi(e) + \frac{2}{3} \left(-\frac{17599\pi}{35} \psi_n(e) - \frac{2268\eta\pi}{5} \zeta_n(e) - \frac{788\pi e^2}{(1-e^2)^2} \varphi_e \right) \right) x \right. \\
& \left. + \frac{64}{18375} \left(-116761\kappa + (19600\pi^2 - 59920C - 59920 \log(4) - 89880 \log(x)) F(e) \right) x^{3/2} \right]. \tag{A7}
\end{aligned}$$

We have derived analytical relations for the various functions that appear in Equation (A7):

$$\phi(e) = \sum_{p=1}^{\infty} \frac{p^3}{4} \left[\left(\left(-e^2 - \frac{3}{e^2} + \frac{1}{e^4} + 3 \right) p^2 + \frac{1}{3} - \frac{1}{e^2} + \frac{1}{e^4} \right) J_p(pe)^2 + \left(-3e - \frac{4}{e^3} + \frac{7}{e} \right) p J'_p(pe) J_p(pe) \right. \\ \left. + \left(\left(e^2 + \frac{1}{e^2} - 2 \right) p^2 + \frac{1}{e^2} - 1 \right) J'_p(pe)^2 \right], \quad (\text{A8})$$

$$\tilde{\phi}(e) = \sum_{p=1}^{\infty} \frac{p^2 \sqrt{1-e^2}}{2} \left[\left(-\frac{2}{e^4} - 1 + \frac{3}{e^2} \right) p J_p(pe)^2 + \left(2 \left(e + \frac{1}{e^3} - \frac{2}{e} \right) p^2 - \frac{1}{e} + \frac{2}{e^3} \right) J'_p(pe) J_p(pe) \right. \\ \left. + 2 \left(1 - \frac{1}{e^2} \right) p J'_p(pe)^2 \right], \quad (\text{A9})$$

where the notation $\phi(e)$, $\tilde{\phi}(e)$ has been chosen to coincide with that used in Ref. [28], such that in Equation (A7):

$$\varphi_e = \frac{192}{985} \frac{\sqrt{1-e^2}}{e^2} \left[\sqrt{1-e^2} \phi(e) - \tilde{\phi}(e) \right]. \quad (\text{A10})$$

In order to decrease the computational burden incurred by the numerical evaluation of Eqs. (A8) and (A9), we have derived analytical expressions that reproduce the numerical results up to the twelfth significant figure in the range $e \in [0, 0.7]$. Setting $\mathcal{E} \equiv (1-e^2)^{-1/2}$, we can write our results as follows:

$$\phi(e) = \mathcal{E}^{10} \left\{ 1 + \frac{18970894028}{2649026657} e^2 + \frac{157473274}{30734301} e^4 + \frac{48176523}{177473701} e^6 + \frac{9293260}{3542508891} e^8 - \frac{5034498}{7491716851} e^{10} \right. \\ \left. + \frac{428340}{9958749469} e^{12} \right\}, \quad (\text{A11})$$

$$\tilde{\phi}(e) = \mathcal{E}^7 \left\{ 1 + \frac{413137256}{136292703} e^2 + \frac{37570495}{98143337} e^4 - \frac{2640201}{993226448} e^6 - \frac{4679700}{6316712563} e^8 - \frac{328675}{8674876481} e^{10} \right\}, \quad (\text{A12})$$

$$(\text{A13})$$

In Eq. (A7) C stands for Euler's constant. The functions $F(e)$, ζ_n , ϕ_n given in Ref. [28] depend on the new functions we present in Equations (A14)- (A19). We have constructed these new analytical formulae ensuring that they reproduce the numerical data provided in Ref. [28] with an accuracy better than 0.1% for eccentricity values $e \in [0, 0.7]$:

$$\psi(e) = \mathcal{E}^{12} \left(1 - \frac{185}{21} e^2 - \frac{3733}{99} e^4 - \frac{1423}{104} e^6 \right), \quad (\text{A14})$$

$$\zeta(e) = \mathcal{E}^{12} \left(1 + \frac{2095}{143} e^2 + \frac{1590}{59} e^4 + \frac{977}{113} e^6 \right), \quad (\text{A15})$$

$$\kappa(e) = \mathcal{E}^{14} \left(1 + \frac{1497}{79} e^2 + \frac{7021}{143} e^4 + \frac{997}{98} e^6 + \frac{463}{51} e^8 - \frac{3829}{120} e^{10} \right), \quad (\text{A16})$$

$$\tilde{\psi}(e) = \mathcal{E}^9 \left(1 - \frac{2022}{305} e^2 - \frac{249}{26} e^4 - \frac{193}{239} e^6 + \frac{23}{43} e^8 - \frac{102}{463} e^{10} \right), \quad (\text{A17})$$

$$\tilde{\zeta}(e) = \mathcal{E}^9 \left(1 + \frac{1563}{194} e^2 + \frac{1142}{193} e^4 + \frac{123}{281} e^6 - \frac{27}{328} e^8 \right), \quad (\text{A18})$$

$$\tilde{\kappa}(e) = \mathcal{E}^{10} \left(1 + \frac{1789}{167} e^2 + \frac{5391}{340} e^4 + \frac{2150}{219} e^6 - \frac{1007}{320} e^8 + \frac{2588}{189} e^{10} \right). \quad (\text{A19})$$

Regarding the evolution of the orbital eccentricity, we have used the 3PN accurate equations derived in Ref. [28]:

$$M\dot{e} = \dot{e}_{0\text{PN}}x^4 + \dot{e}_{1\text{PN}}x^5 + \dot{e}_{2\text{PN}}x^6 + \dot{e}_{3\text{PN}}x^7 + \dot{e}_{\text{HT}}, \quad (\text{A20})$$

where the $e_{i\text{PN}}$ with $i = 1, 2, 3$ are given by Eqs. (6.19a, 6.19b), (C10, C11) of Ref. [28], and the higher-order hereditary terms \dot{e}_{HT} are given by [28]:

$$\begin{aligned} \dot{e}_{\text{HT}} = & \frac{32}{5} e \eta x^4 \left\{ -\frac{985}{48} \pi x^{3/2} \varphi_e(e) + \pi x^{5/2} \left[\frac{55691}{1344} \psi_e(e) + \frac{19067}{126} \eta \zeta_e(e) \right] \right. \\ & \left. + x^3 \left[\left(\frac{89789209}{352800} - \frac{87419}{630} \ln 2 + \frac{78003}{560} \ln 3 \right) \kappa_e(e) - \frac{769}{96} \left(\frac{16}{3} \pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln \left[\frac{4x^{3/2}}{x_0} \right] \right) F_e(e) \right] \right\}. \end{aligned} \quad (\text{A21})$$

The functions $\psi_e(e)$, $\zeta_e(e)$, $\kappa_e(e)$ and $F_e(e)$ are provided in [28], and also depend on the Eqs. (A14)- (A19) we have derived in this paper. We have verified that, as discussed in [28], the arbitrary length scale x_0 cancels out when adding 3PN terms for the orbital eccentricity evolution.

Appendix B

In Section IID we presented a framework to increase the reliability of the ax model to describe binaries with asymmetric mass-ratios. This new prescription is given by Equation (18). We have derived the following coefficients for this expression:

$$\begin{aligned} a_4 = & -5 \eta \alpha_0 - \frac{97\eta^4}{3888} - \frac{18929389\eta^3}{435456} - \frac{3157\pi^2\eta^2}{144} + \frac{54732199\eta^2}{93312} - \frac{47468}{315} \eta \log(x) - \frac{31495\pi^2\eta}{8064} - \frac{856\gamma\eta}{315} \\ & + \frac{59292668653\eta}{838252800} - \frac{1712}{315} \eta \log(2) + \frac{124741 \log(x)}{8820} - \frac{361\pi^2}{126} + \frac{124741\gamma}{4410} + \frac{3959271176713}{25427001600} - \frac{47385 \log(3)}{1568} \\ & + \frac{127751 \log(2)}{1470}, \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} a_{9/2} = & \frac{9731\pi\eta^3}{1344} + \frac{42680611\pi\eta^2}{145152} + \frac{205\pi^3\eta}{6} - \frac{51438847\pi\eta}{48384} - \frac{3424}{105} \pi \log(x) - \frac{6848\gamma\pi}{105} + \frac{343801320119\pi}{745113600} \\ & - \frac{13696}{105} \pi \log(2), \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} a_5 = & \frac{155\alpha_0\eta^2}{12} + \frac{1195\alpha_0\eta}{336} - 6\eta\alpha_1 - \frac{11567\eta^5}{62208} + \frac{51474823\eta^4}{1741824} + \frac{9799\pi^2\eta^3}{384} - \frac{9007776763\eta^3}{11757312} + \frac{216619}{189} \eta^2 \log(x) \\ & - \frac{126809\pi^2\eta^2}{3024} - \frac{2354\gamma\eta^2}{945} + \frac{1362630004933\eta^2}{914457600} - \frac{4708}{945} \eta^2 \log(2) + \frac{53963197\eta \log(x)}{52920} + \frac{14555455\pi^2\eta}{217728} \\ & + \frac{3090781\gamma\eta}{26460} - \frac{847101477593593\eta}{228843014400} - \frac{15795\eta \log(3)}{3136} + \frac{2105111\eta \log(2)}{8820} - \frac{5910592 \log(x)}{1964655} - \frac{21512\pi^2}{1701} \\ & - \frac{11821184\gamma}{1964655} + \frac{29619150939541789}{36248733480960} + \frac{616005 \log(3)}{3136} - \frac{107638990 \log(2)}{392931}, \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} a_{11/2} = & -20\pi\eta\alpha_0 + \frac{49187\pi\eta^4}{6048} - \frac{7030123\pi\eta^3}{13608} - \frac{112955\pi^3\eta^2}{576} + \frac{1760705531\pi\eta^2}{290304} - \frac{189872}{315} \pi \eta \log(x) \\ & - \frac{26035\pi^3\eta}{16128} - \frac{3424\gamma\pi\eta}{315} - \frac{2437749208561\pi\eta}{4470681600} - \frac{6848}{315} \pi \eta \log(2) + \frac{311233\pi \log(x)}{11760} + \frac{311233\gamma\pi}{5880} \\ & + \frac{91347297344213\pi}{81366405120} - \frac{142155}{784} \pi \log(3) + \frac{5069891\pi \log(2)}{17640}, \end{aligned} \quad (\text{B4})$$

$$\begin{aligned}
a_6 = & -\frac{535\alpha_0\eta^3}{36} + \frac{7295\alpha_0\eta^2}{336} - \frac{248065\alpha_0\eta}{4536} + \frac{31\alpha_1\eta^2}{2} + \frac{239\alpha_1\eta}{56} - 7\alpha_2\eta - 7\alpha_3\eta\log(x) - \alpha_3\eta - \frac{155377\eta^6}{1679616} \\
& - \frac{152154269\eta^5}{10450944} - \frac{1039145\pi^2\eta^4}{62208} + \frac{76527233921\eta^4}{94058496} - \frac{41026693\eta^3\log(x)}{17010} + \frac{55082725\pi^2\eta^3}{217728} - \frac{2033\gamma\eta^3}{1701} \\
& - \frac{56909847373567\eta^3}{7242504192} - \frac{4066\eta^3\log(2)}{1701} - \frac{271237829\eta^2\log(x)}{127008} + \frac{92455\pi^4\eta^2}{1152} - \frac{4061971769\pi^2\eta^2}{870912} - \frac{21169753\gamma\eta^2}{317520} \\
& + \frac{3840832667727673\eta^2}{55477094400} - \frac{57915\eta^2\log(3)}{12544} - \frac{2724535\eta^2\log(2)}{21168} - \frac{4387}{63}\pi^2\eta\log(x) - \frac{12030840839\eta\log(x)}{37721376} + \frac{410\pi^4\eta}{9} \\
& - \frac{8774}{63}\gamma\pi^2\eta + \frac{206470485307\pi^2\eta}{1005903360} + \frac{362623282541\gamma\eta}{94303440} - \frac{12413297162366594971\eta}{271865501107200} + \frac{3016845\eta\log(3)}{12544} \\
& - \frac{17548}{63}\pi^2\eta\log(2) + \frac{701463800861\eta\log(2)}{94303440} + \frac{366368\log^2(x)}{11025} + \frac{2930944\log(2)\log(x)}{11025} - \frac{13696}{315}\pi^2\log(x) \\
& + \frac{1465472\gamma\log(x)}{11025} - \frac{155359670313691\log(x)}{157329572400} - \frac{27392\zeta(3)}{105} - \frac{256\pi^4}{45} - \frac{27392\gamma\pi^2}{315} + \frac{1414520047\pi^2}{2619540} + \frac{1465472\gamma^2}{11025} \\
& - \frac{155359670313691\gamma}{78664786200} + \frac{1867705968412371074441833}{154211174411374080000} + \frac{5861888\log^2(2)}{11025} - \frac{37744140625\log(5)}{260941824} - \frac{63722699919\log(3)}{112752640} \\
& - \frac{54784}{315}\pi^2\log(2) + \frac{5861888\gamma\log(2)}{11025} - \frac{206962178724547\log(2)}{78664786200},
\end{aligned} \tag{B5}$$

where the coefficients α_i with $i = 0, 1, 2, 3$ are given by [75]:

$$\alpha_0 = 153.8803, \tag{B6}$$

$$\alpha_1 = -55.13, \tag{B7}$$

$$\alpha_2 = 588, \tag{B8}$$

$$\alpha_3 = -1144. \tag{B9}$$

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